Norwegian University of Science and Technology
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TMA4110 Calculus 3
Fall 2012

Solutions to exercise set 6

$\boxed{1}$ We have $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$, and the task is to determine whether the set is linearly independent or linearly dependent. It is linearly independent if and only if the equation $a\mathbf{v_1} + b\mathbf{v_2} + c\mathbf{v_3} = 0$, where a, b, c are scalars, has only the trivial solution $a = b = c = 0$.

We reduce the augmented matrix of the vector equation:

$$\begin{bmatrix} 1 & 3 & 5 & 0 \\ 2 & -1 & -4 & 0 \\ -3 & 0 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

c is a free variable, and the system has infinitely many solutions. For instance, if we chose $c = -1$, we get $a = 1$ and $b = -2$, and we can write $\mathbf{v_3}$ as $\mathbf{v_3} = \mathbf{v_1} - 2\mathbf{v_2}$. The set is linearly dependent.

$\boxed{2}$ The standard matrix of T is of the form $[T(\mathbf{e_1})\ \ T(\mathbf{e_2})\ \ T(\mathbf{e_3})\ \ T(\mathbf{e_4})]$. We compute

$$T(\mathbf{e_3}) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad T(\mathbf{e_2}) = \begin{bmatrix} 4 \\ 1 \\ 9 \end{bmatrix} \quad T(\mathbf{e_3}) = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \quad T(\mathbf{e_4}) = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$$

Hence the standard matrix is

$$A = \begin{bmatrix} 1 & 4 & -1 & 3 \\ 0 & 1 & 2 & -1 \\ 2 & 9 & 0 & 5 \end{bmatrix}$$

This can also be done “by inspection”, as in Example 5 on page 77.

$\boxed{3}$ T is the linear transformation from Problem 2, so we have the standard matrix

$$A = \begin{bmatrix} 1 & 4 & -1 & 3 \\ 0 & 1 & 2 & -1 \\ 2 & 9 & 0 & 5 \end{bmatrix}$$

According to Theorem 12, page 77, T is onto if and only if the columns of A span $\mathbb{R}^3$. This happens if and only if A has a pivot position in each row. We reduce A and get

$$A = \begin{bmatrix} 1 & 4 & -1 & 3 \\ 0 & 1 & 2 & -1 \\ 2 & 9 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -1 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A does not have a pivot position in the third row, and hence the columns does not span $\mathbb{R}^3$. Then T is not onto.

$\boxed{4}$ We draw a picture of the migration (C means city and S means suburbs):

$$\begin{tikzcd} C \arrow[loop left, "0.90"] \arrow[r, shift left, "0.10"] & S \arrow[l, shift left, "0.20"] \arrow[loop right, "0.80"] \end{tikzcd}$$

The migration matrix is given by

$$M = \begin{bmatrix} 0.90 & 0.20 \\ 0.10 & 0.80 \end{bmatrix}$$

and the population in 2010 is given by

$$\mathbf{x_0} = \begin{bmatrix} 200{,}000 \\ 100{,}000 \end{bmatrix}$$

We are supposed to find $\mathbf{x_2}$, i.e. the population vector for 2012. We have that

$$\mathbf{x_1} = M\mathbf{x_0} = \begin{bmatrix} 0.90 & 0.20 \\ 0.10 & 0.80 \end{bmatrix} \cdot \begin{bmatrix} 200{,}000 \\ 100{,}000 \end{bmatrix} = \begin{bmatrix} 200{,}000 \\ 100{,}000 \end{bmatrix}$$

The number of people living in each region does not change from one year to the next, and hence $\mathbf{x_2} = \mathbf{x_1} = \mathbf{x_0}$.

(See pages 84–85 in the book for more details.)