Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4110 Calculus 3
Fall 2012

Solutions to exercise set 8

$\boxed{1}$ Instead of computing $\det A$ directly (either by the formula for 3×3-matrices (2) on page 164, or by cofactor expansion), we reduce A to get a triangular matrix. This is achieved by subtracting the first row from the last.

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & -1 & 2 \\ 3 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 & 4 \\ 0 & -1 & 2 \\ 0 & 0 & -3 \end{bmatrix} = B$$

From Theorem 3 on page 169, we know that $\det B = \det A$, and since B is triangular, $\det B$ is the product of the diagonal entries. That is,

$$\det A = \det B = 3 \cdot (-1) \cdot (-3) = 9$$

$\boxed{2}$ We use the same method as in the previous exercise. When reducing A to a triangular matrix, we observe that the only types of row operation needed are replacement operations (no scaling or swapping of rows). Then, as in the above exercise, the determinant remains unchanged.

$$A = \begin{bmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 5 & 4 & -3 \\ -3 & -7 & -5 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & -1 & -2 & 5 \\ 0 & 2 & 4 & -10 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B$$

Since B has zero rows, $\det B = 0$, and hence $\det A = 0$.

$\boxed{3}$ We have the matrix

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

To find the inverse of A, we can either use the method from chapter 2, or we can use the inverse formula described in Theorem 8, page 179. The theorem says that

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

Since the third row of A has only one non-zero element, we find $\det A$ by cofactor expansion along this row:

$$\det A = 1 \cdot \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 5$$

The matrix adj A is the transposed cofactor matrix of A. That is, we need to compute *all* the cofactors. We have

$$\operatorname{adj} A^T = \begin{bmatrix} \begin{vmatrix} -2 & 1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 3 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 0 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 \\ 3 & 0 & -1 \\ 7 & 5 & -4 \end{bmatrix}$$

So A^{-1} is

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A = \frac{1}{5} \begin{bmatrix} -1 & 3 & 7 \\ 0 & 0 & 5 \\ 2 & -1 & -4 \end{bmatrix}$$

$\boxed{4}$ We look at the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$. This triangle, which we call S, has area $1/2$.

Let $A = [\mathbf{x} \quad \mathbf{y}] = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$.

Now, the triangle with vertices $\mathbf{0}$, $\mathbf{x}$ and $\mathbf{y}$ is the image of S under the linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ given by A. (To see this, note that the vertices $(1,0)$ and $(0,1)$ of S should go to the vertices (x_1, x_2) and (y_1, y_2) of the new triangle.)

According to Theorem 10 on page 182 (and its generalization on page 183), we know that the area of $T(S)$ – that is, the area we want to compute – is equal to $|\det(A)| \cdot \{\text{Area of } S\}$.

Since the area of S is $1/2$, the area of $T(S)$ is

$$|\det A| \cdot \frac{1}{2} = \frac{|x_1 y_2 - y_1 x_2|}{2}$$