TMA4110 Calculus 3
Fall 2016

Øving 3

4.1 Second-Order Equations

For each of the second-order differential equations in Exercises 1–8, decide whether the equation is linear or nonlinear. If the equation is linear, state whether the equation is homogeneous or inhomogeneous.

1. $y'' + 3y' + 5y = 3\cos 2t$
2. $t^2 y'' = 4y' - \sin t$
3. $t^2 y'' + (1-y)y' = \cos 2t$
4. $ty'' + (\sin t)y' = 4y - \cos 5t$
5. $t^2 y'' + 4yy' = 0$
6. $y'' + 4y' + 7y = 3e^{-t}\sin t$
7. $y'' + 3y' + 4\sin y = 0$
8. $(1-t^2)y'' = 3y$

In Exercises 13 and 14, show, by direct substitution, that the given functions $y_1(t)$ and $y_2(t)$ are solutions of the given differential equation. Then verify, again by direct substitution, that any linear combination $C_1 y_1(t) + C_2 y_2(t)$ of the two solutions is also a solution.

13. $y'' - y' - 6y = 0, \quad y_1(t) = e^{3t}, \quad y_2(t) = e^{-2t}$
14. $y'' + 4y = 0, \quad y_1(t) = \cos 2t, \quad y_2(t) = \sin 2t$

In Exercise 17 and 19, use Definition 1.22 to explain why $y_1(t)$ and $y_2(t)$ are linearly independent solutions of the given differential equation. In addition calculate the Wronskian and use it to explain the independence of the given solutions.

17. $y'' - y' - 2y = 0, \quad y_1(t) = e^{-t}, \quad y_2(t) = e^{2t}$
19. $y'' + 4y' + 13y = 0, \quad y_1(t) = e^{-2t}\cos 3t, \quad y_2(t) = e^{-2t}\sin 3t$

21. (Optional extra) Show that the functions

$$y_1(t) = t^2 \quad \text{and} \quad y_2(t) = t|t|$$

are linearly independent on $(-\infty, \infty)$. Next, show that the Wronskian of the two functions is identically zero on the interval $(-\infty, \infty)$. Why doesn't this result contradict Proposition 1.27?

26. (Optional extra) Unfortunately, Theorem 1.23 does not show us how to find two independent solutions. However, there is a technique that can be used to find a second solution when one solution is known.

(a) Show that $y_1(t) = t^2$ is a solution of

$$t^2 y'' + t y' - 4y = 0. \tag{1}$$

(b) Let $y_2(t) = v y_1(t) = v t^2$, where v is a yet to be determined function of t. Note that if $y_2/y_1 = v$ and v is nonconstant, then y_1 and y_2 are independent. Show that the substitution $y_2 = vt^2$ reduces equation (1) to the separable equation

$$5v' + tv'' = 0. \tag{2}$$

Solve equation (2) for v, form the solution $y_2 = vt^2$, and then state the general solution of equation (1).

4.3 Linear, Homogeneous Equations with Constant Coefficients

The equations in Exercises 1 and 2 have distinct, real, characteristic roots. Find the general solution in each case.

1. $y'' - y' - 2y = 0$
2. $2y'' - 3y' - 2y = 0$

The equations in Exercises 9 and 10 have complex characteristic roots. Find the general solution in each case.

9. $y'' + y = 0$
10. $y'' + 4y = 0$

The equations in Exercises 17 and 18 have repeated, real, characteristic roots. Find the general solution in each case.

17. $y'' - 4y' + 4y = 0$
18. $y'' - 6y' + 9y = 0$

In Exercise 25 and 29, find the solution of the given initial value problem.

25. $y'' - y' - 2y = 0, \quad y(0) = -1, \quad y'(0) = 2$
29. $y'' + 10y' + 25y = 0, \quad y(0) = 2, \quad y'(0) = -1$

38. (Optional extra) Given that the characteristic equation $\lambda^2 + p\lambda + q = 0$ has a double root, $\lambda = \lambda_1$, show, by direct substitution, that $y = te^{\lambda_1 t}$ is a solution of $y'' + py' + qy = 0$.

4.4 Harmonic Motion

In Exercise 7 and 9, place each equation in the form $y = Ae^{-ct}\cos(\omega t - \phi)$. Then, on one plot, place the graph of $y = Ae^{-ct}\cos(\omega t - \phi)$, $y = Ae^{-ct}$, and $y = -Ae^{-ct}$. For the last two, use a different line style and/or color that for the first.

7. $y = e^{-t/2}(\cos 5t + \sin 5t)$
9. $y = e^{-0.1t}(0.2\cos 2t + 0.1\sin 2t)$

13. The undamped system

$$\frac{2}{5}x'' + kx = 0, \quad x(0) = 2, \quad x'(0) = v_0$$

is observed to have period $\pi/2$ and amplitude 2. Find k and v_0.

14. Consider the undamped oscillator

$$mx'' + kx = 0, \quad x(0) = x_0, \quad x'(0) = v_0.$$

Show that the amplitude of the resulting motion is $\sqrt{x_0^2 + mv_0^2/k}$.

21. (Optional extra) If $\mu > 2\sqrt{km}$, the system $mx'' + \mu x' + kx = 0$ is over-damped. The system is allowed to come to equilibrium. Then the mass is given a sharp tap, imparting an instantaneous downward velocity v_0.

(a) Show that the position of the mass is given by

$$x(t) = \frac{v_0}{\gamma} e^{-\mu t/(2m)} \sinh \gamma t,$$

where

$$\gamma = \frac{\sqrt{\mu^2 - 4mk}}{2m}.$$

(b) Show that the mass reaches its lowest point at

$$t = \frac{1}{\gamma} \tanh^{-1} \frac{2m\gamma}{\mu},$$

a time independent initial conditions.

(c) Show that, in the critically damped case, the time it takes the mass to reach its lowest point is given by $t = 2m/\mu$.