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2.1 Matrix Operations

1. Compute each of the following matrix sum or product if it is defined. If an expression is undefined, explain why. Compute $-2A$, $B-2A$, AC, and CD, where

$$A=\begin{bmatrix}2 & 0 & -1\\ 4 & -5 & 2\end{bmatrix},\quad B=\begin{bmatrix}7 & -5 & 1\\ 1 & -4 & -3\end{bmatrix},\quad C=\begin{bmatrix}1 & 2\\ -2 & 1\end{bmatrix},\quad D=\begin{bmatrix}3 & 5\\ -1 & 4\end{bmatrix}.$$

10. Let $A=\begin{bmatrix}2 & -3\\ -4 & 6\end{bmatrix}$, $B=\begin{bmatrix}8 & 4\\ 5 & 5\end{bmatrix}$, and $C=\begin{bmatrix}5 & -2\\ 3 & 1\end{bmatrix}$. Verify that $AB=AC$ and yet $B\neq C$.

11. Let $A=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 4 & 5\end{bmatrix}$ and $D=\begin{bmatrix}2 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 5\end{bmatrix}$. Compute AD and DA. Explain how the columns or rows of A change when A is multiplied by D on the right or on the left. Find a 3×3 matrix, not the identity matrix or the zero matrix, such that $AB=BA$.

29. (Optional Extra) Prove Theorem 2(b) and 2(c). Use the row-column rule. The (i,j)-entry in $A(B+C)$ can be written as

$$a_{i1}(b_{1j}+c_{1j})+\cdots+a_{in}(b_{nj}+c_{nj})\quad\text{or}\quad\sum_{k=1}^{n}a_{ik}(b_{kj}+c_{kj})$$

33. (Optional Extra) Prove Theorem 3(d). (*Hint:* Consider the jth row of $(AB)^T$.)

2.2 The Inverse of a Matrix

25. (Optional Extra) Prove the following for $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$. Show that if $ad-bc=0$, then the equation $A\mathbf{x}=\mathbf{0}$ has more than one solution. Why does this imply that A is not invertible? (*Hint:* First consider $a=b=0$. Then, if a and b are not both zero, consider the vector $\mathbf{x}=\begin{bmatrix}-b\\ a\end{bmatrix}$.)

31. Find the inverses of the matrices if they exist. Use the algorithm introduced in this section.

$$\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$$

24. (Optional Extra) Suppose A is $n \times n$ and the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^n$. Explain why A must be invertible. (*Hint:* Is A row equivalent to I_n?)

2.3 Characterizations of Invertible Matrices

Determine which of the matrices in Exercises **2** and **7** (Optional Extra) are invertible. Use as few calculations as possible. Justify you answers.

2.

$$\begin{bmatrix} -4 & 6 \\ 6 & -9 \end{bmatrix}$$

7. (Optional Extra)

$$\begin{bmatrix} -1 & -3 & 0 & 1 \\ 3 & 5 & 8 & -3 \\ -2 & -6 & 3 & 2 \\ 0 & -1 & 2 & 1 \end{bmatrix}$$

11. The matrices in this Exercise are all $n \times n$. Each part of the exercise has an *implication* of the form "If "statement 1" then "statement 2"." Mark an implication as True if the truth of "statement 2" *always* follows whenever "statement 1" happens to be true. An implication is False if there is an instance which "statement 2" is false but "statement 1" is true. Justify each answer.

a. If the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then A is row equivalent to the $n \times n$ identity matrix.

b. If the columns of A span $\mathbb{R}^n$, then the columns are linearly independent.

c. If A is an $n \times n$ matrix, then the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$

d. If the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, then A has fewer then n pivot positions.

e. If A^T is not invertible, then A is not invertible.

13. (Optional Extra) An $m \times n$ **upper triangular matrix** is one whose entries *below* the main diagonal are 0's (as in Exercise 8). When is a square upper triangular matrix invertible? Justify your answer.

2.5 Matrix Factorization

In Exercises **1** and **4** (Optional Extra), solve the equation $A\mathbf{x} = \mathbf{b}$ by using the LU factorization given for A. In Exercise **1**, also solve $A\mathbf{x} = \mathbf{b}$ by ordinary row reduction.

1.

$$A = \begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -7 \\ 5 \\ 2 \end{bmatrix}$$
$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

4. (Optional Extra)

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix}$$
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -5 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{bmatrix}$$

9. Find an LU factorization of the matrices (with L unit lower triangular).

$$\begin{bmatrix} 3 & -1 & 2 \\ -3 & -2 & 10 \\ 9 & -5 & 6 \end{bmatrix}$$

24. (Optional Extra) (*QR Factorization*) Suppose $A = QR$, where Q and R are $n \times n$, R is invertible and upper triangular, and Q has the property that $Q^T Q = I$. Show that for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution. What computations with Q and R will produce the solution.

3.1 Introduction to Determinants

2. Compute the determinants using a cofactor expansion across the first row. Also compute the determinant by a cofactor expansion down the second column.

$$\begin{vmatrix} 0 & 4 & 1 \\ 5 & -3 & 0 \\ 2 & 3 & 1 \end{vmatrix}$$

Compute the determinants in Exercises **9** and **12** (Optional Extra) by a cofactor expansion. At each step, choose a row or column that involves the least amount of computation.

9.

$$\begin{vmatrix} 4 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 3 & 0 & 0 & 0 \\ 8 & 3 & 1 & 7 \end{vmatrix}$$

12. (Optional Extra)

$$\begin{vmatrix} 3 & 0 & 0 & 0 \\ 7 & -2 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ 3 & -8 & 4 & -3 \end{vmatrix}$$

37. Let $A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$. Write $5A$. Is $\det 5A = 5 \det A$?

38. (Optional Extra) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and let k be a scalar. Find a formula that relates $\det kA$ to k and $\det A$.

3.2 Properties of Determinants

10. Find the determinant by row reduction to echelon form.

$$\begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -2 & -6 \\ -2 & -6 & 2 & 3 & 10 \\ 1 & 5 & -6 & 2 & -3 \\ 0 & 2 & -4 & 5 & 9 \end{vmatrix}$$

40. Let A and B be 4×4 matrices, with $\det A = -3$ and $\det B = -1$. Compute:

a. $\det AB$

b. $\det B^5$

c. $\det 2A$

d. $\det A^T BA$

e. $\det B^{-1}AB$

33. (Optional Extra) Let A and B be square matrices. Show that even though AB and BA may not be equal, it is always true that $\det AB = \det BA$.