



#### 4.4 Coordinate Systems

In Exercise 3 and 4, find the vector  $\mathbf{x}$  determined by the given coordinate vector  $[\mathbf{x}]_{\mathcal{B}}$  and the given basis  $\mathcal{B}$ .

3.

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ -7 \\ 0 \end{bmatrix} \right\}, \quad [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$$

4.

$$\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -7 \\ 3 \end{bmatrix} \right\}, \quad [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -4 \\ 8 \\ -7 \end{bmatrix}$$

7. Find the coordinate vector  $[\mathbf{x}]_{\mathcal{B}}$  for  $\mathbf{x}$  relative to the given basis  $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ .

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -3 \\ 4 \\ 9 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 8 \\ -9 \\ 6 \end{bmatrix}$$

21. (Optional Extra) Let  $\mathbf{B} = \left\{ \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ 9 \end{bmatrix} \right\}$ . Since the coordinate mapping determined by  $\mathcal{B}$  is a linear transformation from  $\mathbb{R}^2$  into  $\mathbb{R}^2$ , this mapping must be implemented by some  $2 \times 2$  matrix  $A$ . Find it. (*Hint:* Multiplication by  $A$  should transform a vector  $\mathbf{x}$  into its coordinate vector  $[\mathbf{x}]_{\mathcal{B}}$ .)

#### 4.5 The Dimension of a Vector Space

3. For each subspace in the exercise, (a) find a basis, and (b) state the dimension.

$$\left\{ \begin{bmatrix} 2c \\ a-b \\ b-3c \\ a+2b \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

9. Find the dimension of the subspace of all vectors in  $\mathbb{R}^3$  whose first and third entries are equal.

12. Find the dimension of the subspace spanned by the given vectors.

$$\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -8 \\ 6 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 7 \end{bmatrix}$$

13. Determine the dimensions of  $Nul A$  and  $Col A$ .

$$A = \begin{bmatrix} 1 & -6 & 9 & 0 & -2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

21. The first four Hermite polynomials are  $1$ ,  $2t$ ,  $-2+4t^2$ , and  $-12t+8t^3$ . These polynomials arise naturally in the study of certain important differential equations in mathematical physics. Show that the first four Hermite polynomials form a basis for  $\mathbb{P}_3$ .

26. (Optional Extra) Let  $H$  be a  $n$ -dimensional subspace of an  $n$ -dimensional space  $V$ . Show that  $H = V$ .

#### 4.6 Rank

3. Assume that the matrix  $A$  is row equivalent to  $B$ . Without calculations, list  $\text{rank} A$  and  $\text{dim} Nul A$ . Then find bases for  $Col A$ ,  $Row A$ , and  $Nul A$ .

$$A = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

5. If a  $3 \times 8$  matrix  $A$  has rank 3, find  $\text{dim} Nul A$ ,  $\text{dim} Row A$ , and  $\text{rank} A^T$ .

13. If  $A$  is a  $7 \times 5$  matrix, what is the largest possible rank of  $A$ ? If  $A$  is a  $5 \times 7$  matrix, what is the largest possible rank of  $A$ ? Explain your answers.

14. (Optional Extra) If  $A$  is a  $4 \times 3$  matrix, what is the largest possible dimension of the row space of  $A$ ? If  $A$  is a  $3 \times 4$  matrix, what is the largest possible dimension of the row space of  $A$ ? Explain.

17.  $A$  is an  $m \times n$  matrix. Mark each statement True or False. Justify each answer.

- a. The row space of  $A$  is the same as the column space of  $A^T$ .
- b. If  $B$  is any echelon form of  $A$ , and if  $B$  has three nonzero rows, then the first three rows of  $A$  form a basis for  $Row A$ .

- c. The dimensions of the row space and the column space of  $A$  are the same, even if  $A$  is not square.
- d. The sum of the dimensions of the row space and the null space of  $A$  equals the number of rows in  $A$ .
- e. On a computer, row operations can change the apparent rank of a matrix.

**28.** (Optional Extra) This Exercise concern an  $m \times n$  matrix  $A$  and what are often called the *fundamental subspaces* determined by  $A$ . Justify the following equalities:

- a.  $\dim \text{Row}A + \dim \text{Nul}A = n$  *Number of columns of  $A$*
- b.  $\dim \text{Col}A + \dim \text{Nul}A^T = m$  *Number of rows of  $A$*

## 4.9 Applications to Markov Chains

**3.** On any given day, a student is either healthy or ill. Of the students who are healthy today, 95% will be healthy tomorrow. Of the students who are ill today, 55% will still be ill tomorrow.

- a. What is the stochastic matrix for this situation?
- b. Suppose 20% of the students are ill on Monday. What fraction or percentage of the students are likely to be ill on Tuesday? On Wednesday?
- c. If a student is well today, what is the probability that he or she will be well two days from now?

**7.** Find the steady-state vector.

$$\begin{bmatrix} .7 & .1 & .1 \\ .2 & .8 & .2 \\ .1 & .1 & .7 \end{bmatrix}$$

**18.** (Optional Extra) Show that every  $2 \times 2$  stochastic matrix has at least one steady-state vector. Any such matrix can be written in the form  $P = \begin{bmatrix} 1 - \alpha & \beta \\ \alpha & 1 - \beta \end{bmatrix}$ , where  $\alpha$  and  $\beta$  are constants between 0 and 1. (There are two linearly independent steady-state vectors if  $\alpha = \beta = 0$ . Otherwise, there is only one.)