TMA4110 Calculus 3

Fall 2016

Øving 12

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5.7 Application to Differential Equations

1. A particle moving in a planar force field has a position vector $\mathbf{x}$ that satisfies $\mathbf{x}' = A\mathbf{x}$. The 2×2 matrix A has eigenvalues 4 and 2, with corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Find the position of the particle at time t assuming that $\mathbf{x}(0) = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$.

3. Solve the initial value problem $\mathbf{x}'(t) = A\mathbf{x}(t)$ for $t \geq 0$, with $\mathbf{x}(0) = (3, 2)$. Classify the nature of the origin as an attractor, repeller, or saddle point of the dynamic system described by $\mathbf{x}' = A\mathbf{x}$. Find the direction of greatest attraction and/or repulsion. If the origin is a saddle point, sketch typical trajectories.

$$A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$$

9. Construct the general solution of $\mathbf{x}' = A\mathbf{x}$ involving complex eigenfunctions and then obtain the general real solution. Describe the shapes of typical trajectories.

$$A = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix}$$

6.1 Inner Product, Length, and Orthogonality

6. Compute

$$\left(\frac{\mathbf{x} \cdot \mathbf{w}}{\mathbf{x} \cdot \mathbf{x}} \right) \mathbf{x}, \quad \text{where,} \quad \mathbf{w} = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$$

In Exercises **9** and **10**, find a unit vector in the direction of the given vector.

9. $\begin{bmatrix} -30 \\ 40 \end{bmatrix}$

10. $\begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}$

Determine which pairs of vectors in Exercises **15** and **16** are orthogonal.

15. a = $\begin{bmatrix} 8 \\ -5 \end{bmatrix}$, **b** = $\begin{bmatrix} -2 \\ -3 \end{bmatrix}$

16. u = $\begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}$, **v** = $\begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$

26. Let **u** = $\begin{bmatrix} 5 \\ -6 \\ 7 \end{bmatrix}$, and let W be the set of all **x** in $\mathbb{R}^3$ such that **u** · **x** = 0. What theorem in Chapter 4 can be used to show that W is a subspace of $\mathbb{R}^3$? Describe W in geometric language.

29. (Optional Extra) Let $W = \text{Span}\{\mathbf{v}_1, \ldots \mathbf{v}_p\}$. Show that if **x** is orthogonal to each $\mathbf{v}_j$, for $1 \leq j \leq p$, the **x** is orthogonal to every vector in W.

Extra (Optional Extra) Let **u** = $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, **v** = $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$, and **w** = $\begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix}$. Find a nonzero linear combination of **u** and **v** that is orthogonal to **w**. (*Hint:* this exercise is very similar to problem 6 of fall 2015 exam)

6.2 Orthogonal Sets

7. Show that $\{\mathbf{u}_1, \mathbf{u}_2\}$ us an orthogonal basis for $\mathbb{R}^2$. Then express **x** as a linear combination of the **u**'s.

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 6 \\ 4 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 9 \\ -7 \end{bmatrix}$$

11. Compute the orhtogonal projection of $\begin{bmatrix} 1 \\ 7 \end{bmatrix}$ onto the line through $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ and the origin.

14. Let **y** = $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and **u** = $\begin{bmatrix} 7 \\ 1 \end{bmatrix}$. Write **y** as the sum of a vector in Span$\{\mathbf{u}\}$ and a vector orthogonal to **u**.

15. Let **y** = $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and **u** = $\begin{bmatrix} 8 \\ 6 \end{bmatrix}$. Compute the distance from **y** to the line through **u** and the origin.

26. (Optional Extra) Suppose W is a subspace of $\mathbb{R}^n$ spanned by n nonzero orthogonal vectors. Explain why $W = \mathbb{R}^n$.

27. (Optional Extra) Let U be a square matrix with orthogonal columns. Explain why U is invertible. (Mention the theorems you use).