TMA4110 Calculus 3
Fall 2016

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6.3 Orthogonal Projections

1. You may assume that $\{\mathbf{u}_1, \ldots, \mathbf{u}_4\}$ is an orthogonal basis for $\mathbb{R}^4$.

$$\mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \\ -4 \\ -1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 3 \\ 5 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -4 \end{bmatrix}, \quad \mathbf{u}_4 = \begin{bmatrix} 5 \\ -3 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 10 \\ -8 \\ 2 \\ 0 \end{bmatrix}$$

Write $\mathbf{x}$ as the sum of two vectors, one in Span$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and the other in Span$\{\mathbf{u}_4\}$.

8. Let W be the subspace spanned by the $\mathbf{u}$'s, and write $\mathbf{y}$ as the sum of a vector in W and a vector orthogonal to W.

$$\mathbf{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$$

11. Find the closest point to $\mathbf{y}$ in the subspace W spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$

$$\mathbf{y} = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

6.4 The Gram-Schmidt Process

9. Find an orthogonal basis for the column space of the following matrix.

$$\begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}$$

13. The columns of Q were obtained by applying the Gram-Schmidt process to the columns of A. Find an upper triangular matrix R such that $A = QR$. Check your work.

$$A = \begin{bmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -5 \\ 1 & 5 \end{bmatrix}, \quad Q = \frac{1}{6}\begin{bmatrix} 5 & -1 \\ 1 & 5 \\ -3 & 1 \\ 1 & 3 \end{bmatrix}$$

15. Find a QR factorization of the following matrix.

$$\begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix}$$

19. Suppose $A = QR$, where Q is $m \times n$ and R is $n \times n$. Show that if the columns of A are linearly independent, then R must be invertible. (*Hint:* Study the equation $R\mathbf{x} = \mathbf{0}$ and use the fact that $A = QR$).

22. Let $\mathbf{u}_1, \ldots, \mathbf{u}_p$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $T(\mathbf{x}) = \text{proj}_W \mathbf{x}$. Show that T is a linear transformation.

6.5 Least-Squares Problems

2. Find a least-square solution of $A\mathbf{x} = \mathbf{b}$ by (a) constructing the normal equations for $\hat{\mathbf{x}}$ and (b) solving for $\hat{\mathbf{x}}$.

$$A = \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix}$$

6. Describe al least-squares solutions of the equation $A\mathbf{x} = \mathbf{b}$.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 7 \\ 2 \\ 3 \\ 6 \\ 5 \\ 4 \end{bmatrix}$$

15. Use the factorization of $A = QR$ to find the least-squares solution of $A\mathbf{x} = \mathbf{b}$.

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ 1/3 & -2/3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$$

19. Let A ne an $m \times n$ matrix. Use the steps below to show that a vector $\mathbf{x}$ in $\mathbb{R}^n$ satisfies $A\mathbf{x} = \mathbf{0}$ if and only if $A^T A\mathbf{x} = \mathbf{0}$. This will show that $\text{Nul}A = \text{Nul}A^T A$.

(a) Show that if $A\mathbf{x} = \mathbf{0}$, then $A^T A\mathbf{x} = \mathbf{0}$.

(b) Suppose $A^T A\mathbf{x} = \mathbf{0}$. Explain why $\mathbf{x}^T A^T A\mathbf{x} = \mathbf{0}$, and use this to show that $A\mathbf{x} = \mathbf{0}$.

20. Let A be an $m \times n$ matrix such that $A^T A$ is invertible. Show that the columns of A are linearly independent. (*Careful:* You may not assume that A is invertible; it may not even be square).

6.6 Applications to Linear Models

2. Find the equation $y = \beta_0 + \beta_1 x$ of the least squares line that best fits the given data points; $(1, 0), (2, 1), (4, 2), (5, 3)$.

7. A certain experiment produces the data $(1, 1.8), (2, 2.7), (3, 3.4), (4, 3.8), (5, 3.9)$. Describe the model that produces a least-square fit of these points by a function of the form

$$y = \beta_1 x + \beta_2 x^2.$$

Such a function might arise, for example, as the revenue from the sale of x units of a product, when the amount offered for sale affects the price to be set for the product.

(a) Give the design matrix, the observation vector, and the unknown parameter vector.

11. According to Kepler's first law, a comet should have an elliptic, parabolic, or hyperbolic orbit (with gravitational attractions from the planets ignored). In suitable polar coordinates, the position (r, ϑ) of a comet satisfies an equation of the form

$$r = \beta + e(r \cdot \cos \vartheta)$$

where β is a constant and e is the *eccentricity* of the orbit, with $0 \geq e < 1$ for an ellipse, $e = 1$ for a parabola, and $e > 1$ for a hyperbola. Suppose observations of a newly discovered comet provides the data below. Determine the type of orbit, and predict where the comet will be when $\vartheta = 4.6$ (radians).

ϑ	0.88	1.10	1.42	1.77	2.14
r	3.00	2.30	1.65	1.25	1.01

7.1 Diagonalization of Symmetric Matrices

9. Determine if the following matrix is orthogonal. If it is, find the inverse.

$$\begin{bmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{bmatrix}$$

22. Orthogonal diagonalize the following matrix, giving an orthogonal matrix P and a diagonal matrix D. The eigenvalues of the matrix are 3 and 5.

$$\begin{bmatrix} 4 & 0 & 1 & 0 \\ 0 & 4 & 0 & 1 \\ 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 4 \end{bmatrix}$$

27. Show that if A is an $n \times n$ symmetric matrix, then $(A\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (A\mathbf{y})$ for all $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$.

36. Let B be an $n \times n$ symmetric matrix such that $B^2 = B$. Any such matrix is called a **projection matrix** (or an **orthogonal projection matrix**). Given any $\mathbf{y}$ in $\mathbb{R}^n$, let $\hat{\mathbf{y}} = B\mathbf{y}$ and $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$.

(a) Show that $\mathbf{z}$ is orthogonal to $\hat{\mathbf{y}}$.

(b) Let W be the column space of B. Show that $\mathbf{y}$ is the sum of a vector in W and a vector in $W^{\perp}$. Why does this prove that $B\mathbf{y}$ is the orthogonal projection of $\mathbf{y}$ onto the column space of B?

7.2 Quadratic Forms

1. Compute the quadratic form $\mathbf{x}^T A\mathbf{x}$, when $A = \begin{bmatrix} 5 & 1/3 \\ 1/3 & 1 \end{bmatrix}$ and

$$(a)\ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (b)\ \mathbf{x} = \begin{bmatrix} 6 \\ 1 \end{bmatrix} \quad (c)\ \mathbf{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

4. Find the matrix of the quadratic form. Assume $\mathbf{x}$ is in $\mathbb{R}^2$.

(a) $5x_1^2 + 16x_1x_2 - 5x_2^2$

(b) $2x_1x_2$

5. Find the matrix of the quadratic form. Assume $\mathbf{x}$ is in $\mathbb{R}^3$.

(a) $3x_1^2 + 2x_2^2 - 5x_3^2 - 6x_1x_2 + 8x_1x_3 - 4x_2x_3$

(b) $6x_1x_2 + 4x_1x_3 - 10x_2x_3$