NTNU
Norwegian University of Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4110 Calculus 3**

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Examination date: 1 December 2016

Examination time (from–to): 9:00–13:00

Permitted examination support material: Specific calculator (Casio fx-82ES Plus, Citizen SR-270X, Citizen SR-270X College or HP 30s), *Matematisk formelsamling* (K. Rottmann)

Other information:
In the grading, each of the ten points counts equally.

You should demonstrate how you arrive at the answers (e.g. by including intermediate answers or by referring to theory or examples from the reading list).

Language: English

Number of pages: 2

Number of pages enclosed: 0

Checked by:

Informasjon om trykking av eksamensoppgave	
Originalen er:	
1-sidig ☐	**2-sidig** ☒
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skal ha flervalgskjema ☐	

Date Signature

Problem 1

a) Find the general solution of the differential equation $y'' + y = 0$.

b) Find the general solution of the differential equation $y'' + y = \sin 2t + t^2 + 1$.

Problem 2

Let V be the real vector space of all polynomials in one variable, x, of degree less than or equal to 2, that is, $V = \{a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$. Define a function $T\colon V \to V$ given by $T(f(x)) = (x+1)f'(x) + f(x)$ for all polynomials $f(x)$ i V, where f' is the derivative of f. Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

a) Show that $\mathcal{B} = \{1, x, x^2\}$ is a basis for V. Show that T is a linear transformation. Show that $[T(f(x))]_{\mathcal{B}} = A[f(x)]_{\mathcal{B}}$ for all $f(x)$ in V, where $[g(x)]_{\mathcal{B}}$ is the coordinate vector of $g(x)$ in V with respect to $\mathcal{B}$.

b) Find the dimension of the column space of A. Decide if A is invertible.

Problem 3

Let $A = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{8} \\ \frac{1}{2} & 0 & \frac{7}{8} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$.

a) Find the eigenvalues of A and a basis for each of the eigenspaces. Is A diagonalisable?

b) A is the stochastic matrix of a Markov chain. Find the steady-state vector for A.

Problem 4

Find the general solution of the system $\mathbf{x}'(t) = A\mathbf{x}(t)$ of differential equations, where

$$A = \begin{bmatrix} -\frac{13}{6} & \frac{5}{6} & \frac{1}{2} \\ -\frac{1}{3} & -\frac{1}{3} & 1 \\ \frac{1}{6} & -\frac{5}{6} & -\frac{5}{2} \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix},$$

and x_1, x_2 and x_3 are differentiable real functions of one real variable. Find a particular solution such that

$$\mathbf{x}(0) = \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}.$$

What happens to this solution when $t \to \infty$?

It is given that

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \qquad \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 0 \\ 3 \\ -5 \end{bmatrix}$$

are eigenvectors of A, corresponding to eigenvalues -1, -2 og -2, respectively.

Problem 5

a) Show that

$$\operatorname{Re}\Big(\frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}\Big) = \frac{1}{2}\Big(1 + \frac{\sin((n+\frac{1}{2})\theta)}{\sin\frac{\theta}{2}}\Big),$$

where $0 < \theta < 2\pi$. (Hint: Multiply the numerator and the denominator of the fraction of which we want the real part with $e^{-i\theta/2}$.)

b) Show that

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2}\Big(1 + \frac{\sin((n+\frac{1}{2})\theta)}{\sin\frac{\theta}{2}}\Big)$$

for $0 < \theta < 2\pi$. (Hint: You can use the formula $1 + z + z^2 + \cdots + z^n = (1 - z^{n+1})/(1 - z)$, where $z \neq 1$, for a finite geometric series.)

Problem 6

Let A be an $n \times n$ matrix. Show that A is symmetric and positive definite if and only if there exists an invertible $n \times n$ matrix such that $A = B^{\mathrm{T}}B$.

(A symmetric matrix A being positive definite means that $\mathbf{x}^{\mathrm{T}}A\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$. This is equivalent to all eigenvalues of A being positive.)