Norwegian University of Science and Technology
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Examination in TMA4110/TMA4115 Calculus 3, August 2013
Solution

Problem 1 Given the matrix $A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ -1 & 2 & 8 & 4 \\ 2 & 1 & 9 & 2 \end{bmatrix}$.

a) Write the solution set of the matrix equation $A\mathbf{x} = \begin{bmatrix} 2 \\ -8 \\ 1 \end{bmatrix}$ in parametric vector form.

Solution. We begin by reducing the augmented matrix of the equation to its reduced echelon form.

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 2 \\ -1 & 2 & 8 & 4 & -8 \\ 2 & 1 & 9 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 & 2 \\ 0 & 2 & 10 & 4 & -6 \\ 0 & 1 & 5 & 2 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 & 2 \\ 0 & 1 & 5 & 2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that the last column does not contain a pivot position, so the equation is consistent. We furthermore see that the first and the second columns contain pivot position, and that the third and forth do not, so we choose x_3 and x_4 to be free variables. The reduced echelon form of the augmented matrix of the equation corresponds to the system

$$\begin{aligned} x_1 + 2x_3 &= 2 \\ x_2 + 5x_3 + 2x_4 &= -3 \end{aligned}$$

It follows that the complete solution of the equation is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 - 2s \\ -3 - 5s - 2t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ -5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

where s and t are free parameters.

b) Find an orthonormal basis for the row space $\mathrm{Row}(A)$.

Solution. We begin by reducing A to its reduced echelon form.

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ -1 & 2 & 8 & 4 \\ 2 & 1 & 9 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 10 & 4 \\ 0 & 1 & 5 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that the first and second rows contain pivot position, and that the third does not. It follows that the first row $\mathbf{v}_1 = \begin{bmatrix} 1 & 0 & 2 & 0 \end{bmatrix}$ of the reduced echelon form of A and the second row $\mathbf{v}_2 = \begin{bmatrix} 0 & 1 & 5 & 2 \end{bmatrix}$ of the reduced echelon form of A form a basis of $\mathrm{Row}(A)$.

We next use the Gram-Schmidt process in order to get an orthogonal basis of $\mathrm{Row}(A)$. Let $\mathbf{u}_1 = \mathbf{v}_1 = \begin{bmatrix} 1 & 0 & 2 & 0 \end{bmatrix}$ and

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 = \begin{bmatrix} 0 & 1 & 5 & 2 \end{bmatrix} - \frac{10}{5}\begin{bmatrix} 1 & 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 1 & 2 \end{bmatrix}.$$

Then $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal basis of $\mathrm{Row}(A)$.

Finally, we normalize $\mathbf{u}_1$ and $\mathbf{u}_2$ in order to get an orthonormal basis of $\mathrm{Row}(A)$. Let $\mathbf{x}_1 = \frac{1}{\|\mathbf{u}_1\|}\mathbf{u}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 0 & 2 & 0 \end{bmatrix}$ and $\mathbf{x}_2 = \frac{1}{\|\mathbf{u}_2\|}\mathbf{u}_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} -2 & 1 & 1 & 2 \end{bmatrix}$. Then $\{\mathbf{x}_1, \mathbf{x}_2\}$ is an orthonormal basis for the row space $\mathrm{Row}(A)$.

Problem 2

a) Diagonalize the matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ (that is, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$).

Solution. The characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3 \\ 1 & 3-\lambda \end{vmatrix} = (1-\lambda)^2 - 3^2 = \lambda^2 - 2\lambda - 8,$$

and the eigenvalues of A are

$$\lambda = \frac{2 \pm \sqrt{2^2 + 4 \cdot 8}}{2} = \frac{2 \pm 6}{2} = \begin{cases} 4 \\ -2 \end{cases}.$$

$A - 4I = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}$. It follows that $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue 4.

$A + 2I = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$. It follows that $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue -2.

Thus, if we let $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$, then $A = PDP^{-1}$.

b) Make a change of variable, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = B \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, that transforms the quadratic form $x_1^2 + 6x_1x_2 + x_2^2$ into a quadratic form with no cross-product term (that is, find a matrix B such that if $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = B \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, then $x_1^2 + 6x_1x_2 + x_2^2 = ay_1^2 + by_2^2$ for a suitable choice of constants a and b).

Solution. The idea is to let B be an orthogonal matrix whose columns are eigenvectors of A. The eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ we found in **a)** are orthogonal to each other so we just have to normalize them. Since $\|\mathbf{v}_1\| = \|\mathbf{v}_2\| = \sqrt{2}$, it follows that if we let $B = \frac{1}{\sqrt{2}}P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, then B is an orthonormal matrix, so $B^T = B^{-1}$ (in fact, $B^T = B^{-1} = B$). Thus, if we let $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = B \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}(y_1 + y_2) \\ \frac{1}{\sqrt{2}}(y_1 - y_2) \end{bmatrix}$, then

$$\begin{aligned} x_1^2 + 6x_1x_2 + x_2^2 &= \left[\frac{1}{\sqrt{2}}(y_1 + y_2)\right]^2 + 6\frac{1}{\sqrt{2}}(y_1 + y_2)\frac{1}{\sqrt{2}}(y_1 - y_2) + \left[\frac{1}{\sqrt{2}}(y_1 - y_2)\right]^2 \\ &= \frac{1}{2}y_1^2 + y_1y_2 + \frac{1}{2}y_2^2 + 3y_1^2 - 3y_2^2 + \frac{1}{2}y_1^2 - y_1y_2 + \frac{1}{2}y_2^2 = 4y_1^2 - 2y_2^2. \end{aligned}$$

(We could also just had let $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = P \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 + y_2 \\ y_1 - y_2 \end{bmatrix}$, then

$$\begin{aligned} x_1^2 + 6x_1x_2 + x_2^2 &= (y_1 + y_2)^2 + 6(y_1 + y_2)(y_1 - y_2) + (y_1 - y_2)^2 \\ &= y_1^2 + 2y_1y_2 + y_2^2 + 6y_1^2 - 6y_2^2 + y_1^2 - 2y_1y_2 + y_2^2 = 8y_1^2 - 4y_2^2. \end{aligned}$$

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Problem 3 Let $y_1(t) = t$ and $y_2(t) = t\ln(t)$ be two solutions of the differential equation

$$t^2y'' - ty' + y = 0$$

on the interval $(0, \infty)$.

a) Show that the set $\{y_1(t), y_2(t)\}$ is a fundamental set of solutions for the above equation on the interval $(0, \infty)$.

Solution. Let us first check that y_1 and y_2 are solutions to the differential equation $t^2y'' - ty' + y = 0$ (this is not really necessary, since it is stated in the problem that they are). We have that $y_1'(t) = 1$ and that $y_1''(t) = 0$, so $t^2y_1''(t) - ty_1'(t) + y_1(t) = 0 - t + t = 0$, and $y_2'(t) = \ln(t) + 1$, $y_2''(t) = 1/t$, so $t^2y_2''(t) - ty_2'(t) + y_2(t) = t - t\ln(t) - t + t\ln(t) = 0$. Thus, y_1 and y_2 are solutions to the differential equation $t^2y'' - ty' + y = 0$.

To show that $\{y_1(t), y_2(t)\}$ is a fundamental set of solutions for the equation, we compute the Wronskian of y_1 and y_2.

$$W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = t(\ln(t) + 1) - t\ln(t) = t \neq 0$$

for all $t \in (0, \infty)$, so $\{y_1(t), y_2(t)\}$ is a fundamental set of solutions for the differential equation $t^2y'' - ty' + y = 0$.

Alternatively, we could show directly that y_1 and y_2 are linearly independent on the interval $(0, \infty)$, for suppose that c_1 and c_2 are constants such that $c_1y_1(t) + c_2y_2(t) = 0$ for all $t \in (0, \infty)$, then we get by evaluating at $t = 1$ that $c_1 = 0$ (since $y_1(1) = 1$ and $y_2(1) = 0$), and then by evaluating at $t = \exp(1)$ that $c_2 = 0$ (since $y_2(\exp(1)) = \exp(1) \neq 0$). This shows that y_1 and y_2 are linearly independent on the interval $(0, \infty)$, and it follows that $\{y_1(t), y_2(t)\}$ is a fundamental set of solutions for the differential equation $t^2y'' - ty' + y = 0$.

b) Find the general solution to the differential equation $t^2y'' - ty' + y = t$, where $0 < t < \infty$.

Solution. We will first find a particular solution by using variations of parameters. So we define a function y on the interval $(0, \infty)$ by letting $y(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$ for each $t \in (0, \infty)$ where v_1 and v_2 are twice differentiable functions to be determined below. Then

$$y'(t) = v_1'(t)y_1(t) + v_2'(t)y_2(y) + v_1(t)y_1'(t) + v_2(t)y_2'(t).$$

We will assume that $v_1'(t)y_1(t) + v_2'(t)y_2(y) = 0$ for all $t \in (0, \infty)$. Then $y'(t) = v_1(t)y_1'(t) + v_2(t)y_2'(t)$, $y''(t) = v_1'(t)y_1'(t) + v_1(t)y_1''(t) + v_2'(t)y_2'(t) + v_2(t)y_2''(t)$, and

$$\begin{aligned}
t^2y''(t) - ty'(t) + y(t) &= t^2(v_1'(t)y_1'(t) + v_1(t)y_1''(t) + v_2'(t)y_2'(t) + v_2(t)y_2''(t)) \\
&\quad - t(v_1(t)y_1'(t) + v_2(t)y_2'(t)) + v_1(t)y_1(t) + v_2(t)y_2(t) \\
&= v_1(t)(t^2y_1''(t) - ty_1'(t) + y_1(t)) + v_2(t)(t^2y_2''(t) - ty_2'(t) + y_2(t)) \\
&\quad + v_1'(t)t^2y_1'(t) + v_2'(t)t^2y_2'(t) \\
&= v_1'(t)t^2 + v_2'(t)t^2(\ln(t) + 1).
\end{aligned}$$

Thus, if $v_1'(t)y_1(t) + v_2'(t)y_2(y) = v_1'(t)t + v_2'(t)t\ln(t) = 0$ and $v_1'(t)t^2 + v_2'(t)t^2(\ln(t)+1) = t$ for all $t \in (0, \infty)$, then y is a solution to the differential equation $t^2y'' - ty' + y = t$ on the interval $(0, \infty)$.

It is easy to see that for any $t \in (0, \infty)$, the system

$$\begin{aligned} v_1'(t)t^2 + v_2'(t)t^2(\ln(t)+1) &= t \\ v_1'(t)t + v_2'(t)t\ln(t) &= 0 \end{aligned}$$

has a unique solution and that this solution is $v_2'(t) = 1/t$ and $v_1'(t) = -\ln(t)/t$.

It follows that if we let

$$v_1(t) = \int v_1'(t)dt = -\int \ln(t)/t\ dt = -\frac{1}{2}(\ln(t))^2,$$

and

$$v_2(t) = \int v_2'(t)dt = \int 1/t\ dt = \ln(t),$$

then

$$y(t) = y_1(t)v_1(t) + y_2(t)v_2(t) = -\frac{1}{2}t(\ln(t))^2 + t(\ln(t))^2 = \frac{1}{2}t(\ln(t))^2$$

is a solution to the differential equation $t^2y'' - ty' + y = t$ on the interval $(0, \infty)$. It follows that the general solution to the differential equation $t^2y'' - ty' + y = t$ on the interval $(0, \infty)$ is

$$y(t) = \frac{1}{2}t(\ln(t))^2 + c_1t + c_2t\ln(t)$$

where c_1 and c_2 are constants.

Problem 4

a) The matrix $A = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -2 & 0 \\ 0 & 2 & -1 \end{bmatrix}$ has 3 eigenvalues. One eigenvalue is $-2+i$, and $\begin{bmatrix} 1+i \\ 1-i \\ -2 \end{bmatrix}$ is an eigenvector corresponding to $-2+i$. Find the other two eigenvalues and a corresponding eigenvector for each of these eigenvalues.

Solution. Since all the entries of A are real, and $\begin{bmatrix} 1+i \\ 1-i \\ -2 \end{bmatrix}$ is an eigenvector of A corresponding to $-2+i$, it follows by complex conjugation that $\begin{bmatrix} 1-i \\ 1+i \\ -2 \end{bmatrix}$ is an eigenvector of A corresponding to $-2-i$.

The characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 0 & 1 \\ 1 & -2-\lambda & 0 \\ 0 & 2 & -1-\lambda \end{vmatrix} = (-1-\lambda)(-2-\lambda)(-1-\lambda) + 2$$
$$= -\lambda^3 - 4\lambda^2 - 5\lambda.$$

It follows that 0 is an eigenvalue of A. In order to find a corresponding eigenvector, we reduce A to an echelon form.

$$\begin{bmatrix} -1 & 0 & 1 \\ 1 & -2 & 0 \\ 0 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 0 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We see that $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue 0.

b) Three tanks contain salt water. Tank 1 holds 10 litres, tank 2 holds 5 litres, and tank 3 holds 10 litres. Salt water flows from tank 1 to tank 2 at a rate of 10 litres per second, and from tank 2 to tank 3 at a rate of 10 litres per second, and from tank 3 to tank 1 at a rate of 10 litres per second. Suppose that initially tank 1 contains 10 grammes of salt, tank 2 contains 6 grammes of salt, and tank 3 contains 4 grammes of salt. Suppose also that the tanks are stirred so that the salt in each tank is evenly distributed. How much salt does each of the three tanks contains after $\pi/2$ seconds?

Solution. Let $x_1(t)$ be the amount of salt in tank 1 at time t, let $x_2(t)$ be the amount of salt in tank 2 at time t, and let $x_3(t)$ be the amount of salt in tank 3 at time t. Then

$$x_1'(t) = -x_1(t) + x_3(t)$$
$$x_2'(t) = x_1(t) - 2x_2(t)$$
$$x_3'(t) = 2x_2(t) - x_3(t)$$

and $x_1(0) = 10$, $x_2(0) = 6$, $x_3(0) = 4$.

Thus, it follows from **a)** that

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} 1+i \\ 1-i \\ -2 \end{bmatrix} e^{(-2+i)t} + c_2 \begin{bmatrix} 1-i \\ 1+i \\ -2 \end{bmatrix} e^{(-2-i)t} + c_3 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

where c_1, c_2 and c_3 are constants satisfying

$$c_1 \begin{bmatrix} 1+i \\ 1-i \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 1-i \\ 1+i \\ -2 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \\ 4 \end{bmatrix}$$

To solve the last equation, we reduce the augmented matrix of the equation to its reduced echelon form.

$$\begin{bmatrix} 1+i & 1-i & 2 & 10 \\ 1-i & 1+i & 1 & 6 \\ -2 & -2 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} -2 & -2 & 2 & 4 \\ 1-i & 1+i & 1 & 6 \\ 1+i & 1-i & 2 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 2i & 2-i & 8-2i \\ 0 & -2i & 3+i & 12+2i \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 2i & 2-i & 8-2i \\ 0 & 0 & 5 & 20 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 2i & -i & -2i \\ 0 & 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 2 & -1 & -2 \\ 0 & 0 & 1 & 4 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

Thus $c_1 = c_2 = 1$ and $c_3 = 4$, and

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 1+i \\ 1-i \\ -2 \end{bmatrix} e^{(-2+i)t} + \begin{bmatrix} 1-i \\ 1+i \\ -2 \end{bmatrix} e^{(-2-i)t} + 4 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix} e^{-2t}\cos(t) + \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} e^{-2t}\sin(t) + \begin{bmatrix} 8 \\ 4 \\ 8 \end{bmatrix}$$

It follows that $x_1(\pi/2) = 8 - 2e^{-\pi}$, $x_2(\pi/2) = 4 + 2e^{-\pi}$ and that $x_3(\pi/2) = 8$, so tank 1 contains $8 - 2e^{-\pi} \approx 7.91357$ grammes of salt after $\pi/2$ seconds, tank 2 contains $4 + 2e^{-\pi} \approx 4.08643$ grammes of salt after $\pi/2$ seconds, and tank 3 contains 8 grammes of salt after $\pi/2$ seconds.

Problem 5 *You do not have to give reasons for your answers for this problem.*

a) For each of the following 4 statements, determine whether it is true or not.

1. The equation $x_2 = x_1(2 + \sqrt{2}i)$ is linear.
   True.

2. The differential equation $y'' + 16y = e^{-4t} + 3\sin(4t)$ is linear.
   True.
3. The differential equation $t^2 y''(t) + 3ty'(t) - 3y(t) = 0$ is linear.
   True.
4. The transformation T from $\mathbb{R}$ to $\mathbb{R}$ given by $T(x) = x^2 + x$ is linear.
   False. We have for example that $T(2) = 6 \neq 4 = 2T(1)$.

b) For each of the following 4 statements, determine whether it is true or not.

1. The three lines $2x_1 + x_3 = 1$, $-2x_1 + x_2 = 3$ and $x_1 + x_3 = 1$ have exactly one point in common.
   True. The matrix $\begin{bmatrix} 2 & 0 & 1 & 1 \\ -2 & 1 & 0 & 3 \\ 1 & 0 & 1 & 1 \end{bmatrix}$ can be reduced to $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ so $(0, 3, 1)$ is the only point which belong to all three lines.
2. If $\mathbf{a}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$, then $\mathbf{b}$ belongs to $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$.
   False. The matrix $\begin{bmatrix} 0 & 1 & 4 \\ -1 & -1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ can be reduced to $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ which has a pivot position in the last column, so $\mathbf{b}$ does not belong to $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$.
3. If $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, then $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathbb{R}^4$.
   False. A set of three vectors cannot generate $\mathbb{R}^4$.
4. The following vectors are linearly independent: $\begin{bmatrix} 2 \\ -4 \\ 8 \end{bmatrix}$, $\begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$.
   False. $\begin{bmatrix} 2 \\ -4 \\ 8 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$.

c) For each of the following 4 statements, determine whether it is true or not.

1. If A is an $m \times n$ matrix, B is an $n \times m$ matrix, and $AB = 0$, then we must either have that $A = 0$ or that $B = 0$.
   False. We have for example that $\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$.

2. If C and D are $n \times n$-matrices, then it must be the case that $(C+D)(C-D) = C^2 - D^2$.
   False. We have for example that

$$\left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\right)\left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} - \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\right) = \begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\-1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}$$

   and

$$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}^2 - \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}^2 = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} - \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$$

3. If E is an invertible matrix, then $(2E)^{-1} = 2E^{-1}$.
   False. $(2E)^{-1} = \frac{1}{2}E^{-1}$.

4. If F is an 2×2 matrix and the equation $F\mathbf{x} = \begin{bmatrix}1\\2\end{bmatrix}$ has a unique solution, then F must be invertible.
   True. If the equation $F\mathbf{x} = \begin{bmatrix}1\\2\end{bmatrix}$ has a unique solution, then $\mathrm{Nul}(F) = \{0\}$ (because if $\mathbf{u} \in \mathrm{Nul}(F)$, $\mathbf{u} \neq \mathbf{0}$ and $F\mathbf{x} = \begin{bmatrix}1\\2\end{bmatrix}$, then $F(\mathbf{x}+\mathbf{u}) = \begin{bmatrix}1\\2\end{bmatrix}$), and it then follows from the invertible matrix theorem that F is invertible.

d) Let A be an $n \times n$ matrix and k a scalar. For each of the following 4 statements, determine whether it is true or not.

1. $\det(kA) = k^n \det(A)$.
   True. kA can be produced by multiplying each of the n rows of A by k, so it follows from Theorem 3 in Section 3.2 that $\det(kA) = k^n \det(A)$.
2. If $\det(A) = 2$, then $\det(A^2) = 4$.
   True. $\det(A^2) = \det(A)\det(A)$ by Theorem 6 in Section 3.2.
3. $\det(A^T) = \det(A)$.
   True. See Theorem 5 in Section 3.2.
4. If A is invertible, then $\det(A)\det(A^{-1}) = 0$.
   False. $\det(A)\det(A^{-1}) = \det(A)\det(A)^{-1} = 1 \neq 0$.

e) Let V be a vector space different from the zero vector space, and let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ be vectors in V. For each of the following 4 statements, determine whether it is true or not.

1. If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is linearly independent, then it must be the case that $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is a basis for V.
   False. It is only true if $\dim(V) = p$.

2. If $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\} = V$, then some subset of $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is a basis for V.
   True. See Theorem 5 in Section 4.3.
3. If $\dim(V) = p$, then $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ cannot be linearly independent.
   False. If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is a basis for V, then $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is linearly independent.
4. If $\dim(V) = p$, then $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ must be a basis for V.
   False. $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is only a basis for V if it is linearly independent or $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\} = V$.

f) Let A be an $n \times n$ matrix. For each of the following 4 statements, determine whether it is true or not.

1. If A is invertible and 1 is an eigenvalue of A, then 1 must also be an eigenvalue of A^{-1}.
   True. If $A\mathbf{v} = \mathbf{v}$, then $\mathbf{v} = (A^{-1}A)\mathbf{v} = A^{-1}\mathbf{v}$.
2. If $\mathbf{v}$ is an eigenvector of A, then $\mathbf{v}$ must also be an eigenvector of A^2.
   True. If $A\mathbf{v} = \lambda\mathbf{v}$, then $A^2\mathbf{v} = \lambda A\mathbf{v} = \lambda^2\mathbf{v}$.
3. If A has fewer than n distinct eigenvalues, then A cannot be diagonalizable.
   False. 2 is the only eigenvalue of $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, and $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ is a diagonale matrix and therefore diagonalizable.
4. If every vector in $\mathbb{R}^n$ can be written as a linear combination of eigenvectors of A, then A must be diagonalizable.
   True. If every vector in $\mathbb{R}^n$ can be written as a linear combination of eigenvectors of A, then $\mathbb{R}^n$ has a basis consisting of eigenvectors of A, and it then follows from Theorem 5 in Section 5.3 that A is diagonalizable.

g) Let $\mathbf{v}$ and $\mathbf{u}$ be vectors in $\mathbb{R}^n$ and let W be a subspace of $\mathbb{R}^n$. For each of the following 4 statements, determine whether it is true or not.

1. The distance between $\mathbf{v}$ and $\mathbf{u}$ is $\|\mathbf{u} - \mathbf{v}\|$.
   True.
2. If $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent, then $\mathbf{u}$ and $\mathbf{v}$ must be orthogonal to each other.
   False. We have for example that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are linearly independent but not orthogonal to each other.
3. If $\mathbf{v}$ coincides with its orthogonal projection onto W, then $\mathbf{v}$ must belong to W.
   True. The orthogonal projection of $\mathbf{v}$ onto W belongs to W so if it coincides with $\mathbf{v}$, then $\mathbf{v}$ must belong to W.

4. No vector in $\mathbb{R}^n$ can belong to both W and $W^\perp$ (the orthogonal complement of W).
   False. $\mathbf{0}$ belongs to both W and $W^\perp$.

h) Let A be an $n \times n$ matrix. For each of the following 4 statements, determine whether it is true or not.

1. If $A^T = A$ and if $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$ which satisfy that $A\mathbf{u} = 5\mathbf{u}$ and $A\mathbf{v} = 2\mathbf{v}$, then $\mathbf{u}$ and $\mathbf{v}$ must be orthogonal to each other.
   True. See Theorem 1 in Section 7.1.
2. If $A = PDP^T$ where $P^T = P^{-1}$ and D is a diagonal matrix, then A must be symmetric.
   True. $A^T = (PDP^T)^T = PDP^T = A$ because $D^T = D$.
3. If A is symmetric and all the eigenvalues of A are positive, then the quadratic form $\mathbf{x} \to \mathbf{x}^T A\mathbf{x}$ must be positive definite.
   True. See Theorem 5 in Section 7.2.
4. If A is symmetric and λ is an eigenvalue of A, then the dimension of the eigenspace of A corresponding to λ must be equal to the multiplicity of λ as a root of the characteristic polynomial of A.
   True. See Theorem 3 in Section 7.1.