TMA 4115 Matematikk 3

Lecture 11 for MBIOT5, MTKJ, MTNANO

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11. February 2014

Linear sytems and associated equations

Multiplication: Matrix $\times$ **Vector**: For $A = [\overrightarrow{a_1}\ \overrightarrow{a_2}\ \dots\ \overrightarrow{a_k}]$ and $\overrightarrow{x} = (x_1, x_2, \dots, x_k)$ we defined

$$A\overrightarrow{x} = x_1\overrightarrow{a_1} + x_2\overrightarrow{a_2} + \cdots + x_k\overrightarrow{a_k}$$

Note: The product is a linear combination of the columns of A.

Linear system $\leftrightarrow$ vector equation $\sum_{i=1}^{k} r_i\overrightarrow{a_i} = \overrightarrow{b}$
$\leftrightarrow$ matrix equation $A\overrightarrow{x} = \overrightarrow{b}$

To solve the equations apply *Gaussian elimination* to $[A\,\overrightarrow{b}]$

Linear sytems and associated equations II

A linear system is **homogeneous** if we can write it as $A\overrightarrow{x} = \overrightarrow{0}$.

The general solution to $A\overrightarrow{x} = \overrightarrow{b}$ in **parametric form** is:

$$\overrightarrow{x} = \overrightarrow{v_p} + r_1\overrightarrow{v_1} + \ldots + r_k\overrightarrow{v_k}$$

where $\overrightarrow{v_p}$ is a particular solution to $A\overrightarrow{x} = \overrightarrow{b}$ and $\overrightarrow{v_1}, \ldots, \overrightarrow{v_k}$ are the solutions of $A\overrightarrow{x} = \overrightarrow{0}$ associated to the free variables.

Solutions of matrix equations

We were interested in the following questions about $A\overrightarrow{x} = \overrightarrow{b}$

1. Is there a solution at all (i.e. is the system consistent)?
2. Is there a unique solution?

For $A = [\overrightarrow{a_1}\ \overrightarrow{a_2}\ \dots\ \overrightarrow{a_n}]$ question 1 can be rephrased as:

- Is $\overrightarrow{b} \in \text{span}\{\overrightarrow{a_1}, \dots, \overrightarrow{a_n}\}$?

What about the second question?

Idea: The general solution of $A\overrightarrow{x} = \overrightarrow{b}$ will produce a unique solution if $A\overrightarrow{x} = \overrightarrow{0}$ has a unique solution.

The trivial solution $\overrightarrow{0}$ always exists. Hence, $A\overrightarrow{x} = \overrightarrow{0}$ has a unique solution if and only if $\overrightarrow{0}$ is the only solution.

Summary: span and linear (in-)dependence

Let $\overrightarrow{v_1}, \dots, \overrightarrow{v_p}$ be vectors and $A = \begin{bmatrix} \overrightarrow{v_1} & \overrightarrow{v_2} & \dots & \overrightarrow{v_p} \end{bmatrix}$.

- span$\{\overrightarrow{v_1}, \dots, \overrightarrow{v_p}\}$ is the set of all vectors which can be generated from $\overrightarrow{v_1}, \dots, \overrightarrow{v_p}$.

  Test for $\overrightarrow{b} \in$ span$\{\overrightarrow{v_1}, \dots, \overrightarrow{v_p}\}$: If $A\overrightarrow{x} = \overrightarrow{b}$ is consistent, $\overrightarrow{b}$ is in the span.

- Linear (in-)dependence is about redundance, i.e. whether we can combine vectors from fewer vectors.

  Test for linear (in-)dependence of $\{\overrightarrow{v_1}, \dots, \overrightarrow{v_p}\}$:
  If $[A\,\overrightarrow{0}]$ contains only basic variables, the family is linearly independent. Otherwise, the family is linearly dependent.

Moral: Use Gaussian elimination to settle these questions!