TMA 4115 Matematikk 3

Lecture 13 for MBIOT5, MTKJ, MTNANO

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Matrix $A \to$ linear transformation $T_A \colon \mathbb{R}^n \to \mathbb{R}^m, \overrightarrow{x} \mapsto A\overrightarrow{x}$

$T \colon \mathbb{R}^n \to \mathbb{R}^m$ linear $\to$ **standard matrix** $\begin{bmatrix} T(\overrightarrow{e_1}) & \dots & T(\overrightarrow{e_n}) \end{bmatrix}$

Formulate questions about $A\overrightarrow{x} = \overrightarrow{b}$ in the language of linear transformations. Recall $T \colon \mathbb{R}^n \to \mathbb{R}^m$ is

- **onto** if each $\overrightarrow{b} \in \mathbb{R}^m$ is the image of <u>at least</u> one $\overrightarrow{x} \in \mathbb{R}^n$
- **one-to-one** if each $\overrightarrow{b} \in \mathbb{R}^m$ is the image of <u>at most</u> one $\overrightarrow{x} \in \mathbb{R}^n$

(In the literature: **onto** = **surjective** , **one-to-one** = **injective**)

11.11 Theorem

Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if $T(\overrightarrow{x}) = \overrightarrow{0}$ has only the trivial solution $\overrightarrow{0}$.

Proof: If T is one-to-one, there is at most one solution to $T(\overrightarrow{x}) = \overrightarrow{0}$. Hence, only the trivial $\overrightarrow{x} = \overrightarrow{0}$ solves $T(\overrightarrow{x}) = \overrightarrow{0}$.

Conversly let only $\overrightarrow{x} = \overrightarrow{0}$ solve $T(\overrightarrow{x}) = \overrightarrow{0}$.

Assume that $\overrightarrow{u}, \overrightarrow{v} \in \mathbb{R}^n$ satisfy $T(\overrightarrow{v}) = T(\overrightarrow{u})$. Then

$$\overrightarrow{0} = T(\overrightarrow{u}) - T(\overrightarrow{v}) = T(\overrightarrow{u} - \overrightarrow{v})$$

Hence, $\overrightarrow{u} - \overrightarrow{v} = \overrightarrow{0}$ and thus $\overrightarrow{u} = \overrightarrow{v}$. Each vector in $\mathbb{R}^m$ may thus only be the image of at most one vector in $\mathbb{R}^n$. $\square$

Rephrase questions about $A\overrightarrow{x} = \overrightarrow{b}$

Let $A = \begin{bmatrix} \overrightarrow{a}_1 & \dots & \overrightarrow{a}_n \end{bmatrix}$ be an $n \times m$ matrix.

1. Is $A\overrightarrow{x} = \overrightarrow{b}$ consistent for all $\overrightarrow{b} \in \mathbb{R}^m$?
   $\leftrightarrow$ Is every $\overrightarrow{b}$ in the span$\{\overrightarrow{a_1}, \dots, \overrightarrow{a_n}\}$?
   $\leftrightarrow$ Is $T_A \colon \mathbb{R}^n \to \mathbb{R}^m$ onto $\mathbb{R}^m$?
2. Is there a unique solution to $A\overrightarrow{x} = \overrightarrow{b}$ for $\overrightarrow{b} \in \mathbb{R}^m$?
   $\leftrightarrow$ Are $\overrightarrow{a}_1, \dots, \overrightarrow{a}_n$ linearly independent?
   $\leftrightarrow$ Is $T_A \colon \mathbb{R}^n \to \mathbb{R}^m$ one-to-one?

12. Matrix algebra

Let $f, g\colon \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations and $r \in \mathbb{R}$. Then $f + rg\colon \mathbb{R}^n \to \mathbb{R}^m$, $\overrightarrow{x} \mapsto f(\overrightarrow{x}) + rg(\overrightarrow{x})$ is linear:

$$\begin{aligned}(f + rg)(\overrightarrow{v} + t\overrightarrow{u}) &= f(\overrightarrow{u} + t\overrightarrow{v}) + rg(\overrightarrow{u} + t\overrightarrow{v}) \\ &= (f + rg)(\overrightarrow{u}) + t(f + rg)(\overrightarrow{v})\end{aligned}$$

For $h\colon \mathbb{R}^m \to \mathbb{R}^p$ linear the composition $h \circ f\colon \mathbb{R}^n \to \mathbb{R}^p$ is linear.

Question: Are the standard matrices of $(f + rg)$ and $h \circ g$ related to the standard matrices of f, g and h?