TMA 4115 Matematikk 3

Lecture 15 for MBIOT5, MTKJ, MTNANO

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Invertible matrices

A $n \times n$ square matrix A is called **invertible** if there is a square matrix A^{-1} with

$$A \cdot A^{-1} = I_n \quad A^{-1} \cdot A = I_n$$

The equation $A\overrightarrow{x} = \overrightarrow{b}$ for a $n \times n$ matrix A has a unique solution for all $\overrightarrow{b} \in \mathbb{R}^n$ if and only if A is invertible.

Use Gaussian elimination on $\begin{bmatrix} A & I_n \end{bmatrix}$ to compute if A is invertible and if so, get A^{-1}.

Invertible matrices and determinants

Special case 2×2**:** For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we know:

A is invertible if the **determinant** $\det A = ad - bc$ is not zero.

We saw similar behavior for second order differential equations:
The Wronskian was defined as the determinant of a 2×2 matrix.

Question: Can we define a "determinant" for arbitrary square matrices?

The 3×3 case

Use Gaussian elimination to row reduce an (invertible) 3×3 matrix (with $a_{11} \neq 0$):

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightsquigarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11}a_{21} & a_{11}a_{22} & a_{11}a_{23} \\ a_{11}a_{31} & a_{11}a_{32} & a_{11}a_{33} \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31} \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & 0 & a_{11}\Delta \end{bmatrix}$$

$$\text{with } \Delta = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.$$

Now A can only be invertible, if $\Delta \neq 0$.

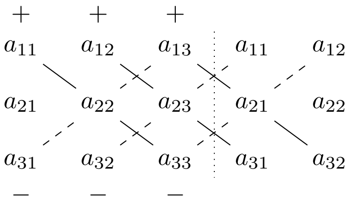
3×3 determinants

We define

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \Delta = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$- a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.$$

If you want to remember this formula, use the “rule of Sarrus”:



Problem: How to produce a general formula for the $n \times n$-case?

Derive a general formula

We rewrite the formula Δ for the 3×3 matrix:

$$\begin{aligned}\Delta =&(a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}) + (a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33}) \\ &+ (a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}) \\ =&a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) \\ &+ a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ =&a_{11}\text{det} \begin{bmatrix} a_{22} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} - a_{12}\text{det} \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\text{det} \begin{bmatrix} a_{21} & a_{22} \\ a_{32} & a_{31} \end{bmatrix}\end{aligned}$$

Delete the i-th row and j-th column in A to form a matrix A_{ij} then

$$\Delta = a_{11}\text{det}A_{11} + a_{12}\text{det}A_{12} + a_{13}\text{det}A_{13}$$

13.1 Recursive Definition of Determinants

Fix a $n \times n$ matrix A. If $1 \leq i, j \leq n$ we form matrices A_{ij} by deleting in A the i-th row and
j-th column.

Define the **determinant** of the matrix A:
For $n = 1$: $\det \begin{bmatrix} a_{11} \end{bmatrix} = a_{11}$.
For $n \geq 2$ and $A = \begin{bmatrix} a_{ij} \end{bmatrix}_{1 \leq i,j \leq n}$ define the determinant as

$$\begin{aligned} \det A &= a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + a_{1n} \det A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j} \end{aligned}$$

Aim of chapter 13:

1. Characterize determinants (and compute them),
2. consider applications of determinants.