TMA 4115 Matematikk 3

Lecture 16 for MBIOT5, MTKJ, MTNANO

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Determinants

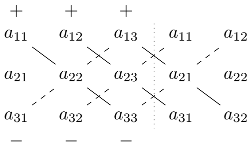
For a square matrix $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ the **determinant** is the number:

$$\begin{aligned} \det A &= a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + a_{1n} \det A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j} \end{aligned}$$

Idea: $\det A \neq 0$ if and only if A is invertible.

Formulae for $n = 2$: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

and for $n = 3$ via Sarrus rule:



Warning: Sarrus rule only works for the 3×3 case!

How to compute determinants

Cofactor expansion: Use only if there are rows or columns with many $0's$.

Theorem The determinant of an upper triangular matrix is the product of the main diagonal elements

Examples (upper triangular matrices):

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad D = \begin{bmatrix} a & 1 & 1 \\ 0 & b & 1 \\ 0 & 0 & c \end{bmatrix}$$

$\det\ A = 2$ $\quad\det\ B = 0$ $\quad\det\ C = 0$ $\quad\det\ D = abc$

Not upper triangular: $\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$

How to compute determinants II

Algorithm with Gaussian elimination:
Compute det A

- Use Gaussian elimination to row reduce A to upper triangular form U, track information:
  $r =$ number of row swaps ,
  row scales $k_1, \dots, k_s$ (scaling operation, *not* adding multiples of different rows).
- det $A = \frac{(-1)^r}{k_1 \cdot k_2 \cdot \ldots \cdot k_s} \cdot$ det U

Moral: If asked to compute a determinant usually use Gaussian elimination.

Example

Compute the determinant of $\begin{bmatrix} 3 & 3 & -3 \\ -2 & -2 & 0 \\ 0 & 2 & 1 \end{bmatrix}$:

$$\begin{vmatrix} 3 & 3 & -3 \\ -2 & -2 & 0 \\ 0 & 2 & 1 \end{vmatrix} = 3 \begin{vmatrix} 1 & 1 & -1 \\ -2 & -2 & 0 \\ 0 & 2 & 1 \end{vmatrix} = 3 \begin{vmatrix} 1 & 1 & -1 \\ 0 & 0 & -2 \\ 0 & 2 & 1 \end{vmatrix}$$

$$= 3(-1) \begin{vmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{vmatrix} = 3(-1)(1 \cdot 2 \cdot -2) = 12$$

Determinants and geometry

13.15 Theorem Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation

- If $n = 2$ and S is a parallelogram in $\mathbb{R}^2$ then

$$\text{area of } T(S) = |\det A_T| \cdot \text{area of } S$$

- If $n = 3$ and S is a paralleliped in $\mathbb{R}^3$ then

$$\text{volume of } T(S) = |\det A_T| \cdot \text{volume of } S$$

General principle (from a multivariable calculus course): If $f\colon \mathbb{R}^n \supseteq U \to \mathbb{R}^d$ and $\phi\colon V \to U$ are sufficiently differentiable (and ϕ satisfies some further properties) then

$$\int_U f(\overrightarrow{x})d\overrightarrow{x} = \int_V f \circ \phi(\overrightarrow{y}) \cdot |\det D\phi| d\overrightarrow{y}$$

Summary

Determinants are an important theoretical tool (not only if we want to solve equations).

However, most jobs which involve matrix computation (e.g. is the matrix invertible, solving linear systems...) can be solved more efficiently by computers without using determinants.

We need determinants as a tool for our theory. If you only want to compute (and have a fast computer) its probably best to avoid determinants.