TMA 4115 Matematikk 3

Lecture 17 for MBIOT5, MTKJ, MTNANO

Alexander Schmeding

NTNU

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An interesting observation

So far we have studied:

Linear differential equations

$$y'' + 2y' + y = g(x)$$

Goal: Find **functions** solving the equation

Linear systems

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ 42x_1 - x_3 &= 12 \\ x_2 + x_3 &= 1 \end{aligned}$$

Goal: Find **numbers/vectors** solving the system

Both topics seem to be connected! (Determinants, general solutions, homogeneous equations...)

Question: What is the theoretical explanation?

Linear transformations

Rewrite the linear system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ 42x_1 - x_3 &= 12 \\ x_2 + x_3 &= 1 \end{aligned}$$

as a matrix equation $A\overrightarrow{x} = \begin{bmatrix} 0 \\ 12 \\ 1 \end{bmatrix}$ and associate a linear transformation $T_A \colon \mathbb{R}^3 \to \mathbb{R}^3$. Solutions to the linear system are solutions to $T_A(\overrightarrow{x}) = \begin{bmatrix} 0 \\ 12 \\ 1 \end{bmatrix}$

Can we view a linear differential equation in the same way?

Linear transformations (on functions?)

The left hand side of $y'' + 2y' + y = g(x)$ is a transformation for functions:

f	$f'' + 2f' + f$
e^x	$4e^x$
$\sin(x)$	$2\cos(x)$
$\cos(x)$	$-2\sin(x)$
t	t
t^2	$2 + 2t + t^2$

It is even a "linear" transformation:

f	$f'' + 2f' + f$
$e^x + \sin(x)$	$4e^x + 2\cos(x)$
$t + t^2 + \cos(x)$	$t + 2 + 2t + t^2 - 2\sin(x)$
0	0

Functions behave in this example like vectors!

Functions as vectors?

Indeed functions $f, g, h\colon \mathbb{R} \to \mathbb{R}$ behave like vectors. We can pointwise add and multiply with real numbers:

- $f + g = g + f$
- $(f + g) + h = f + (g + h)$
- Let 0 be the function which is constant 0, then $0 + f = f = f + 0$
- $r \cdot (s \cdot f) = (rs) \cdot f = s \cdot (r \cdot f)$
- $(r + s)f = r \cdot f + s \cdot f$
- $r \cdot (f + g) = r \cdot f + r \cdot g$
- $f + (-1) \cdot f = f - f = 0$
- $1 \cdot f = f$

14.1 Definition (abstract) vector space

Fix $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A ($\mathbb{K}$-)**vector space** is a non-empty set V of objects, called **vectors**, with operations "+" *addition* and "·" *multiplication* by **scalars** (=numbers in $\mathbb{K}$).
For all vectors $\overrightarrow{u}$, $\overrightarrow{v}$ and $\overrightarrow{w}$ in V and $r, s \in \mathbb{K}$ the following rules must hold:

- $\overrightarrow{u} + \overrightarrow{v} = \overrightarrow{v} + \overrightarrow{u}$
- $(\overrightarrow{u} + \overrightarrow{v}) + \overrightarrow{w} = \overrightarrow{u} + (\overrightarrow{v} + \overrightarrow{w})$
- There is a **zero-vector** $\overrightarrow{0}$ with $\overrightarrow{0} + \overrightarrow{v} = \overrightarrow{v}$
- for $\overrightarrow{v}$ there is $-\overrightarrow{v}$ with $\overrightarrow{v} + (-\overrightarrow{v}) = \overrightarrow{0}$ $((-1) \cdot \overrightarrow{v} = -\overrightarrow{v})$
- $r(\overrightarrow{u} + \overrightarrow{v}) = r\overrightarrow{u} + r\overrightarrow{v}$
- $(r + s)\overrightarrow{v} = r\overrightarrow{v} + s\overrightarrow{v}$
- $(rs)\overrightarrow{v} = r(s\overrightarrow{v})$
- $1\overrightarrow{v} = \overrightarrow{v}$

In this lecture, we consider only $\mathbb{R}$-vector spaces. So if nothing else is said, assume $\mathbb{K} = \mathbb{R}$.

Vector spaces

(Most important) Example: $\mathbb{R}^n$ is a vector space.

If you have trouble with abstract vector spaces, always think of how things work in $\mathbb{R}^n$!

Vector spaces generalize $\mathbb{R}^n$. Thus they show

- what properties in $\mathbb{R}^n$ are important for linear algebra,
- how to extend linear algebra to more general situations.

To showcase the above principles we extend some old definitions:

Familiar concepts now in vector spaces

Let V be a vector space and $\overrightarrow{v_1}, \ldots, \overrightarrow{v_k} \in V$. A **linear combination** of $\overrightarrow{v_1}, \ldots, \overrightarrow{v_k}$ is a weighted sum

$$\sum_{l=1}^{k} r_l \overrightarrow{v_l} = r_1 \overrightarrow{v_1} + \ldots + r_k \overrightarrow{v_k}$$

We define span $\{ \overrightarrow{v_1}, \ldots, \overrightarrow{v_k} \}$, the **set of all linear combinations** of $\overrightarrow{v_1}, \ldots, \overrightarrow{v_k}$.

The vectors $\overrightarrow{v_1}, \ldots, \overrightarrow{v_k}$ are called **linearly independent** if

$$\sum_{i=1}^{k} r_i \overrightarrow{v}_i = r_1 \overrightarrow{v}_1 + r_2 \overrightarrow{v}_2 + \ldots + r_k \overrightarrow{v}_k = \overrightarrow{0}$$

has only the trivial solution $r_1 = r_2 = \cdots = r_k = 0$.