TMA 4115 Matematikk 3

Lecture 18 for MBIOT5, MTKJ, MTNANO

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14.1 Definition (abstract) vector space

Fix $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A ($\mathbb{K}$-)**vector space** is a non-empty set V of objects, called **vectors**, with operations "$+$" *addition* and "$\cdot$" *multiplication* by **scalars** (=numbers in $\mathbb{K}$).

A **subspace** H of V is a subset $H \subseteq V$ such that

- $\overrightarrow{0} \in H$,
- for $\overrightarrow{v}, \overrightarrow{w} \in H$ and $r \in \mathbb{K}$ the sum $\overrightarrow{v} + r\overrightarrow{w} \in H$.

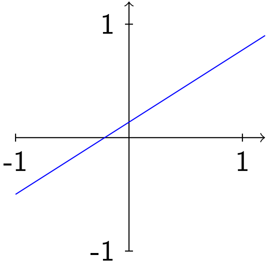
Idea: Vector spaces behave like $\mathbb{R}^n$ and the many important examples arise as subspaces of $\mathbb{R}^n$.

Examples: Subspaces of $\mathbb{R}^2$

- $\mathbb{R}^2, \{\,\overrightarrow{0}\,\}$ are subspaces of $\mathbb{R}^2$.
- For a vector $\overrightarrow{x} \in \mathbb{R}^2$ the set span $\{\,\overrightarrow{x}\,\}$ is a subspace of $\mathbb{R}^2$. If $\overrightarrow{x} \neq \overrightarrow{0}$, span $\{\,\overrightarrow{x}\,\}$ can be drawn as a line through the origin.

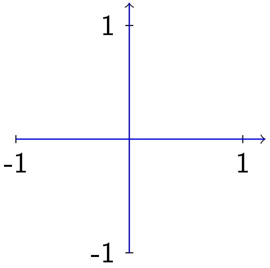
Subsets of $\mathbb{R}^2$ which are not subspaces:

$\{\,(1, -0.5) + r(2.2, 1.4) | r \in \mathbb{R}\,\}$



Does not contain $\overrightarrow{0}$!

$U = \{\,(x, y) \in \mathbb{R}^2 | xy = 0\,\}$



$(1, 0), (0, 1) \in U$ but
$(1, 0) + (0, 1) = (1, 1) \notin U$!

We want to determine how vectors in a subspace can be generated. To this end recall:

The set $\{ \overrightarrow{v_1}, \ldots, \overrightarrow{v_k} \} \subseteq V$ in a vector space V (or shorter the vectors $\overrightarrow{v_1}, \ldots, \overrightarrow{v_k}$) is **linearly independent** if

$$\sum_{i=1}^{k} r_i \overrightarrow{v}_i = r_1 \overrightarrow{v}_1 + r_2 \overrightarrow{v}_2 + \ldots + r_k \overrightarrow{v}_k = \overrightarrow{0}$$

has only the trivial solution $r_1 = r_2 = \cdots = r_k = 0$.

Idea: In $\mathbb{R}^n$ a linear independent set is a "minimal set" which generates a span.

Algorithms to find linearly independent sets

Let $\{ \overrightarrow{v}_1, \ldots, \overrightarrow{v}_k \}$ be a subset of a vector space. If the set is not linearly independent, we want to produce a linearly independent set with the same span. Two strategies:

1. Remove a term which is a linear combination of the other. This does not change the span. Repeat as often as necessary. In the end we obtain a spanning set which is linearly independent.
2. Build the spanning set step by step: Start with a non-zero vector and consider for each vector: "Does this enlarge the span already obtained?" If so, add it. If not, throw it away. This builds up a linearly independent set until it spans.

Second strategy can be done by Gaussian Elimination (cf. chapter on Gaussian elimination).

14.17 Definition Linear transformations

Let V, W be vector spaces. A function $T\colon V \to W$ is called a **linear transformation** if for all vectors $\overrightarrow{v}, \overrightarrow{w}$ and each scalar $r \in \mathbb{K}$ the following holds

$$T(\overrightarrow{u} + r\overrightarrow{v}) = T(\overrightarrow{u}) + rT(\overrightarrow{v})$$

For a linear transformation $T\colon V \to W$ we define

kernel of T: $\ker\, T = \{\, \overrightarrow{v} \in V | T(\overrightarrow{v} = 0\,\}$

image of T: $\mathrm{im}\, T = \{\, \overrightarrow{w} \in W | \exists \overrightarrow{x} \in V \text{ with } T(\overrightarrow{x}) = \overrightarrow{w}\,\}$

Note: $\ker\, T$ is a subspace of V and $\mathrm{im}\, T$ is a subspace of W.

Example: If $V = \mathbb{R}^n, W = \mathbb{R}^m$ then a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation for some matrix A. We see $\ker\, T = \mathrm{Nul}\,(A)$ and $\mathrm{im}\, T = \mathrm{Col}\,(A)$.

More examples for linear transformations

- $S \colon \mathbb{R}^3 \to \mathbb{P}_2$, $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \mapsto a_1 + a_2 t + a_3 t^2$,then
  $\ker S = \{ \overrightarrow{0} \}$ and $\operatorname{im} S = \mathbb{P}_2$
- $\mathrm{ev}_0 \colon C^0(\mathbb{R}, \mathbb{R}) \to \mathbb{R}, f \mapsto f(0)$,then
  $\ker \mathrm{ev}_0 = \{ f \in C^0(\mathbb{R}, \mathbb{R}) | f(0) = 0 \}$ and $\operatorname{im} \mathrm{ev}_0 = \mathbb{R}$.
- $\delta \colon C^2(\mathbb{R}, \mathbb{R}) \to C^0(\mathbb{R}, \mathbb{R}), f \mapsto f'' + \omega^2 f$,then

$$\begin{aligned} \ker \delta &= \left\{ \text{solutions to } f'' + \omega^2 f = 0 \right\} \\ &= \operatorname{span} \{ \cos(\omega t), \sin(\omega t) \} . \\ \operatorname{im} \delta &= \{ g \in C^0(\mathbb{R}, \mathbb{R}) \text{ with } f'' + \omega^2 f = g \\ &\quad \text{for } f \in C^2(\mathbb{R}, \mathbb{R}) \} \end{aligned}$$