TMA 4115 Matematikk 3

Lecture 18 for MBIOT5, MTKJ, MTNANO

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Basis of a vector space

Let V be a vector space. A subset $\mathcal{B} = \{ \overrightarrow{b}_1, \ldots, \overrightarrow{b}_n \}$ is called **basis** of V if

- span $\{ \overrightarrow{b}_1, \ldots, \overrightarrow{b}_n \} = V$
- the set $\mathcal{B}$ is linearly independent

We can think of a basis as a "minimal" system generating V.

Example: The unit vectors $\overrightarrow{e}_1, \ldots \overrightarrow{e}_n \in \mathbb{R}^n$ are a basis for $\mathbb{R}^n$, the standard basis

14.17 Definition Linear transformations

Let V, W be vector spaces. A function $T : V \to W$ is called a **linear transformation** if for all vectors $\overrightarrow{v}, \overrightarrow{w}$ and each scalar $r \in \mathbb{K}$ the following holds

$$T(\overrightarrow{u} + r\overrightarrow{v}) = T(\overrightarrow{u}) + rT(\overrightarrow{v})$$

Example: If $V = \mathbb{R}^n, W = \mathbb{R}^m$ then a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation for some matrix A.

Idea: Use linear transformations to translate problems in abstract vector spaces to $\mathbb{R}^n$.

14.18 Theorem

Let $\mathcal{B} = \{\overrightarrow{b}_1, \ldots, \overrightarrow{b}_n\}$ be a basis of a vector space V then each $\overrightarrow{x} \in V$ can be written as a unique linear combination

$$\overrightarrow{x} = \sum_{i=1}^{n} c_i \overrightarrow{b}_i$$

In particular this gives an (invertible!) linear transformation

$$K_{\mathcal{B}} \colon V \to \mathbb{R}^n, \overrightarrow{x} = \sum_{i=1}^{n} c_i \overrightarrow{b}_i \mapsto (c_1, c_2, \ldots, c_n)$$

Translating problems to $\mathbb{R}^n$

Finding an unknown polynomial

In an experiment we observe the following values of an unknown function f:

time t	0	1	2	3
$f(t)$	.4	1.2	$-.2$	0

Can we approximate f with something simple, i.e. is there a polynomial of (at most) degree 3 which takes these values?

This is a question about the abstract vector space $\mathbb{P}_3$.
Idea: Translate the problem to one in $\mathbb{R}^4$

$$S\colon \mathbb{R}^4 \to \mathbb{P}_3, \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \mapsto \sum_{i=0}^3 a_i t^i \text{ is linear, one-to-one and onto}$$

i.e. we can translate every polynomial to a unique vector in $\mathbb{R}^4$.

Translating problems to $\mathbb{R}^n$

Recall that the maps

$$\text{ev}_k\colon \mathbb{P}_3 \to \mathbb{R}, p(t) \mapsto p(k), k = 0, 1, 2, 3$$

are linear. With their help we rewrite

time t	0	1	2	3
$f(t)$	.4	1.2	$-.2$	0

as

$\text{ev}_0(p) = .4$, $\text{ev}_1(p) = 1.2$, $\text{ev}_2(p) = -.2$ and $\text{ev}_3(p) = 0$.

We are now searching for $\overrightarrow{x} \in \mathbb{R}^4$ such that $\overrightarrow{x}$ satisifes simultaneously the system of equations

$$\begin{aligned}
\text{ev}_0 \circ S(\overrightarrow{x}) &= .4 \\
\text{ev}_1 \circ S(\overrightarrow{x}) &= 1.2 \\
\text{ev}_2 \circ S(\overrightarrow{x}) &= -.2 \\
\text{ev}_3 \circ S(\overrightarrow{x}) &= 0
\end{aligned}$$

Translating problems to $\mathbb{R}^n$

Now $\mathrm{ev}_k \circ S \colon \mathbb{R}^4 \to \mathbb{R}$ are linear maps, whence they are matrix transformations! Their standard matrices compute as

$$A_{\mathrm{ev}_0 \circ S} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$
$$A_{\mathrm{ev}_1 \circ S} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$
$$A_{\mathrm{ev}_2 \circ S} = \begin{bmatrix} 1 & 2 & 4 & 8 \end{bmatrix}$$
$$A_{\mathrm{ev}_3 \circ S} = \begin{bmatrix} 1 & 3 & 9 & 27 \end{bmatrix}$$

And we have to solve the matrix equation

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{bmatrix} \cdot \overrightarrow{x} = \begin{bmatrix} .4 \\ 1.2 \\ -.2 \\ 0 \end{bmatrix}$$

Translating problems to $\mathbb{R}^n$

Solving the matrix equation we obtain $\overrightarrow{x} = \begin{bmatrix} .4 \\ \frac{19}{6} \\ -3 \\ \frac{19}{30} \end{bmatrix}$

Hence $S(\overrightarrow{x}) = .4 + \frac{19}{6}t - 3t^2 + \frac{19}{30}t^3$ solves the problem.