TMA 4115 Matematikk 3

Lecture 20 for MBIOT5, MTKJ, MTNANO

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12. March 2014

Dimension of a vector space

If V is spanned by a finite number of vectors, it is **finite dimensional**.
The number $n \in \mathbb{N}_0$ of vectors in a basis for V is the **dimension** of V and we write $\dim\ V = n$.

Example $\dim\ \mathbb{R}^n = n$

If you want to compute the dimension, construct a basis and see how many elements it has!

14.33 Example

Compute bases for Row (A), Col (A) and Nul (A) where

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & -3 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

Gaussian elimination :

$$A \rightsquigarrow B = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

14.33 Example (cont.)

$$\text{Basis for Row }(A) = \left\{ \begin{bmatrix} 1 \\ 3 \\ -5 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -5 \end{bmatrix} \right\}$$

$$\text{Basis for Col }(A) = \left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$$

Basis for Nul (A): First transform B to reduced echelon form:

$$A \rightsquigarrow B \rightsquigarrow C = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We see for $C\overrightarrow{x} = 0$ the variables x_3 and x_5 are free.

14.33 Example (cont.)

Solving for these variables we get:

$$\text{Basis for Nul }(A) = \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$$

In particular for the 4×5 matrix A we see

$$\dim \text{Row }(A) = 3,\ \dim \text{Col }(A) = 3,\ \dim \text{Nul }(A) = 2$$
$$\dim \text{Col}(A) + \dim \text{Nul }(A) = 5 = \#\text{ Columns of } A$$

Two problems from the exams

Autumn 2008 5b:

Assume that each year 30% of owners of cars with two-wheel drive change to a car with four-wheel drive, whilst 10% of owners of cars with four-wheel drive change to a car with two-wheel drive. The total number of cars is constant, and each car owner has only one car. Given that 25% of car owners have four-wheel drive now, what percentage of car ownwers will have four-wheel drive in ten years' time?

Two problems from the exams II

Spring 2011 6b:

There are two places in Trondheim with bicycles that can be hired for free: Gløshaugen (G) and Torget (T). The bicycles can be hired from early in the morning and must be returned to one of the places the same evening. It is found that of the bicycles hired from G, 80% are returned to G and 20% to T. Of the bicycles hired from T, 30% are returned to G and 70% to T. We assume that this pattern is constant, that all bicycles are hired out each morning, and that no bicycles are stolen. In the long term, what proportion of the bicycles will be at Gløshaugen each morning?

Similarities

- The population is divided into a finite set of mutually exclusive states.
- The system evolves in discrete time intervals and in each interval the individuals can change state.
- An individual changes state according to a set list of probabilities that depends only on the current state and is independent of time.

This situation often occurs when we model (dynamical) systems in the natural sciences!

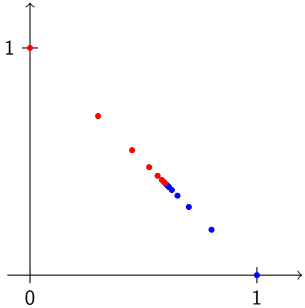
We call such systems **Markov chains**.

15.3 Spring 2011 6 b
Example for a Markov chain

$$P = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix},$$

$\overrightarrow{x}_{i+1} = P\overrightarrow{x_i}, \; i = 1, 2, 3, \dots$

Two initial states $\overrightarrow{x}_0$: $\overrightarrow{e}_1$, $\overrightarrow{e}_2$



i	1	2	3	4	5	6	7
$\overrightarrow{e}_1$	$\begin{bmatrix} .8 \\ .2 \end{bmatrix}$	$\begin{bmatrix} .7 \\ .3 \end{bmatrix}$	$\begin{bmatrix} .65 \\ .35 \end{bmatrix}$	$\begin{bmatrix} .625 \\ .375 \end{bmatrix}$	$\begin{bmatrix} .6125 \\ .3875 \end{bmatrix}$	$\begin{bmatrix} .6062 \\ .3938 \end{bmatrix}$	$\begin{bmatrix} .6031 \\ .3969 \end{bmatrix}$
$\overrightarrow{e}_2$	$\begin{bmatrix} .3 \\ .7 \end{bmatrix}$	$\begin{bmatrix} .45 \\ .55 \end{bmatrix}$	$\begin{bmatrix} .525 \\ .475 \end{bmatrix}$	$\begin{bmatrix} .5625 \\ .4375 \end{bmatrix}$	$\begin{bmatrix} .5812 \\ .4188 \end{bmatrix}$	$\begin{bmatrix} .5906 \\ .4094 \end{bmatrix}$	$\begin{bmatrix} .5953 \\ .4047 \end{bmatrix}$

Apparently for both initial values the system runs towards $\begin{bmatrix} .6 \\ .4 \end{bmatrix}$.