TMA 4115 Matematikk 3

Lecture 21 for MBIOT5, MTKJ, MTNANO

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Rank of a matrix

Let A be a $m \times n$ matrix.
The **rank** of A is the dimension of Col (A).

$$\begin{aligned} \text{rank } (A) &= \text{dim Col } (A) = \text{dim Row } (A) \\ &= \text{dim Col } (A^T) = \text{rank } (A^T) \end{aligned}$$

If A is a $n \times n$ matrix, A is invertible if and only if rank $(A) = n$ (i.e. A is of full rank)

Example of a Markov chain

Spring 2011 6b:

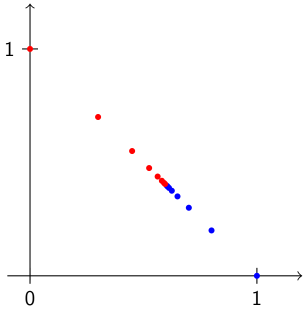
There are two places in Trondheim with bicycles that can be hired for free: Gløshaugen (G) and Torget (T). The bicycles can be hired from early in the morning and must be returned to one of the places the same evening. It is found that of the bicycles hired from G, 80% are returned to G and 20% to T. Of the bicycles hired from T, 30% are returned to G and 70% to T. We assume that this pattern is constant, that all bicycles are hired out each morning, and that no bicycles are stolen. In the long term, what proportion of the bicycles will be at Gløshaugen each morning?

15.3 Spring 2011 6 b
Example for a Markov chain

$P = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$,

$\vec{x}_{i+1} = P\vec{x_i}$, $i = 1, 2, 3, \ldots$

Two initial states $\vec{x}_0$: $\vec{e}_1$, $\vec{e}_2$



i	1	2	3	4	5	6	7
$\vec{e}_1$	$\begin{bmatrix} .8 \\ .2 \end{bmatrix}$	$\begin{bmatrix} .7 \\ .3 \end{bmatrix}$	$\begin{bmatrix} .65 \\ .35 \end{bmatrix}$	$\begin{bmatrix} .625 \\ .375 \end{bmatrix}$	$\begin{bmatrix} .6125 \\ .3875 \end{bmatrix}$	$\begin{bmatrix} .6062 \\ .3938 \end{bmatrix}$	$\begin{bmatrix} .6031 \\ .3969 \end{bmatrix}$
$\vec{e}_2$	$\begin{bmatrix} .3 \\ .7 \end{bmatrix}$	$\begin{bmatrix} .45 \\ .55 \end{bmatrix}$	$\begin{bmatrix} .525 \\ .475 \end{bmatrix}$	$\begin{bmatrix} .5625 \\ .4375 \end{bmatrix}$	$\begin{bmatrix} .5812 \\ .4188 \end{bmatrix}$	$\begin{bmatrix} .5906 \\ .4094 \end{bmatrix}$	$\begin{bmatrix} .5953 \\ .4047 \end{bmatrix}$

Answer to the problem: In the long term the Markov chain converges towards the steady state vector $\begin{bmatrix} .6 \\ .4 \end{bmatrix}$.

The long term behaviour of Markov chains

A stochastic matrix P is called **regular** if there is some $k \in \mathbb{N}$ such that P^k has only strictly positive entries.

Examples:

$$P = \begin{bmatrix} .5 & .25 & .25 \\ 0 & .25 & .25 \\ .5 & .5 & .5 \end{bmatrix} \text{ is regular since } P^2 = \begin{bmatrix} .375 & .3125 & .3125 \\ .125 & .1875 & .1875 \\ .5 & .5 & .5 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ is not regular}$$

The long term behaviour of Markov chains II

15.5 Theorem If P is a regular $n \times n$ stochastic matrix, then P has a unique steady-state vector $\overrightarrow{q}$.
For any intial state $\overrightarrow{x}_0$ the Markov chain $\{x_k\}_{k \in \mathbb{N}_0}$ with $\overrightarrow{x}_{k+1} = P\overrightarrow{x}_k$ converges to $\overrightarrow{q}$ as $k \to \infty$.

To find a steady-state vector:

- Check if the stochastic matrix P is regular
- Compute as in 15.2.
  Do **not** try to compute $P^k \overrightarrow{x}_0$ for $k \to \infty$

Special vectors attached to matrices

The steady-state vectors which determine the long term behaviour of a Markov chain satisfy

$$P\overrightarrow{x} = \overrightarrow{x}$$

Idea: Study matrices using these special kind of vectors.

Vectors which "reproduce" via the following formula

$$A\overrightarrow{x} = \lambda\overrightarrow{x}, \quad \lambda \text{ a scalar}$$

will allow us to discover hidden structures in matrices.