TMA 4115 Matematikk 3

Lecture 22 for MBIOT5, MTKJ, MTNANO

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19. March 2014

Eigenvector and eigenvalue

Let A be a square matrix. A vector $\overrightarrow{x} \neq \vec{0}$ is called **eigenvector**, if

$$A\overrightarrow{x} = \lambda\overrightarrow{x} \quad \text{for some scalar } \lambda$$

Then we call λ an **eigenvalue** of A.

Notice: Eigenvectors must be non-zero, but eigenvalues can be zero:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \cdot \overrightarrow{x} = \overrightarrow{0} = 0\overrightarrow{x}$$

so each $\overrightarrow{x} \in \mathbb{R}^2$ with $\overrightarrow{x} \neq \overrightarrow{0}$ is an eigenvector for the zero matrix, whose only eigenvalue is 0.

Let $\overrightarrow{v}$ and $\overrightarrow{w}$ be eigenvectors associated to the eigenvalue λ and $\mu \neq 0$ a scalar.
If $\overrightarrow{v} + \overrightarrow{w} \neq 0$ then $\overrightarrow{v} + \overrightarrow{w}$ is an eigenvector associated to λ.
Also $\mu\overrightarrow{v}$ is an eigenvector associated to λ.

How to find eigenvalues?

Example: Find the eigenvalues for $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

Calculate the roots of the **characteristic polynomial**

$$\begin{aligned} \det\,(A - \lambda I_2) &= \det\,\left(\begin{bmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix}\right) \\ &= (1-\lambda)^2 - 1 = \lambda^2 - 2\lambda = (\lambda - 0)(\lambda - 2) \end{aligned}$$

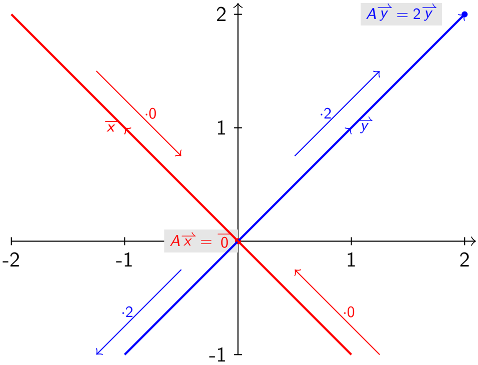
Roots of the characteristic polynomial = eigenvalues.

The roots (=eigenvalues) are $\lambda = 0$ and $\lambda = 2$, both have (algebraic-) multiplicity 1.

After finding the eigenvalues $0, 2$ we calculate eigenvectors. Start for $\lambda = 0$ by Gaussian elimination: Find a non trivial solution of $(A - 0I_2)\overrightarrow{x} = \overrightarrow{0}$: $A - 0I_2 \rightsquigarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

An eigenvector associated to 0 is $\overrightarrow{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Similarly we find the eigenvector associated to 2: $\overrightarrow{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.



Everything along the red line gets multiplied by 0.

the blue line gets multiplied by 2.

Observation

The eigenvectors $\overrightarrow{x}$ and $\overrightarrow{y}$ form a basis $\mathcal{B}$ for $\mathbb{R}^2$.

For a vector $\overrightarrow{x}$ the matrix acts on the $\mathcal{B}$-coordinates $\left[\overrightarrow{x}\right]_{\mathcal{B}}$ by scalar multiplication.

$$\begin{array}{ccc}
\overrightarrow{x} & \xrightarrow{\quad A\cdot \quad} & A\overrightarrow{x} \\
\downarrow & & \downarrow \\
\left[\overrightarrow{x}\right]_{\mathcal{B}} & \xrightarrow{\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \cdot} & \left[A\overrightarrow{x}\right]_{\mathcal{B}}
\end{array}$$

$\rightarrow$ The matrix acts on this basis like a diagonal matrix!

Idea: Can we use eigenvectors to transform the given matrix into a diagonal matrix?

17.4 Diagonalisation of matrices

Let A be a $n \times n$ matrix. We want to find a similar diagonal matrix (if possible). Do the following

1. Compute the eigenvalues of A
2. Compute the eigenvectors associated to the eigenvalues. (Gaussian elimination!)
3. Check if we have n linear independent eigenvectors.
   Hint: Eigenvectors of different eigenvalues are always linearly independent. By construction the eigenvectors produced in 2. are linearly independent. Check if 2. produces n eigenvectors!
4. Order the eigenvectors and write them as columns in a matrix (this is P)
5. Construct a diagonal matrix D whose main diagonal entries are the eigenvalues corresponding to the columns in P.
6. Then $A = PDP^{-1}$