TMA 4115 Matematikk 3

Lecture 23 for MBIOT5, MTKJ, MTNANO

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25. March 2014

Eigenvector and eigenvalue

Let A be a $n \times n$-matrix. **Eigenvector** $\overrightarrow{x}$ with **eigenvalue** λ if

$$A\overrightarrow{x} = \lambda\overrightarrow{x} \quad \text{holds for the scalar } \lambda.$$

The matrix A is called **diagonalizable** if
there is an invertible matrix P and a diagonal matrix D with

$$A = PDP^{-1}.$$

Equivalently: $\mathbb{R}^n$ has a basis of eigenvectors of A

Complex Eigenvalues

A real $n \times n$ matrix A can have complex eigenvalues.
Those always appear as conjugate pairs $\lambda, \overline{\lambda}$.

To find eigenvectors for the eigenvalue λ with $\mathrm{Im}(\lambda) \neq 0$, we let A act on $\mathbb{C}^n$. The associated eigenvector $\overrightarrow{v}$ is a complex vector.

The complex conjugate of $\overrightarrow{v}$ is an eigenvector for $\overline{\lambda}$.

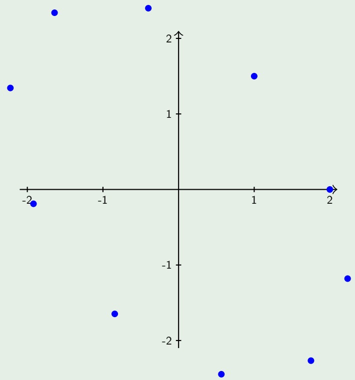
Revisiting example 18.3

$A = \begin{bmatrix} .5 & -.6 \\ .75 & 1.1 \end{bmatrix}$ eigenvalues $.8 \pm .6i$ with eigenvectors $\begin{bmatrix} 2 \pm 4i \\ 5 \end{bmatrix}$.

Does the complex eigenvalue determine the behavior of the action of the matrix on points?

Matrices with complex eigenvalue

What happens if we apply A from 18.3 again and again?
Example point $(2, 0)$:



Observation: The path of the point seems to be an ellipsis.
Is this typical?

18.5 Theorem

Let A be a real 2×2 matrix with complex eigenvalue $\lambda = a + ib$ ($b \neq 0$) and associated eigenvector $\overrightarrow{v} \in \mathbb{C}^2$. Then

$$A = PCP^{-1} \text{ where } P = \left[\mathrm{Re}(\overrightarrow{v})\mathrm{Im}(\overrightarrow{v})\right]$$

$$\text{and } C = \sqrt{a^2 + b^2}\begin{bmatrix}\cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi\end{bmatrix}$$

here φ is the angle between the positive x-axis and the ray from $(0, 0)$ through (a, b), i.e. $\varphi = \mathrm{Arg}\,(\lambda)$

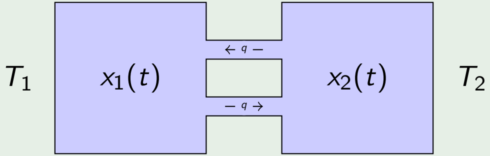
Note: A complex eigenvalue whose imaginary part is not 0 lets A act up to coordinate change as a rotation on $\mathbb{R}^2$.

A generalized version of the theorem holds for $\mathbb{R}^n$.

Another exam problem

Kont 2012, Problem 7

Two water tanks, T_1 and T_2, each with volume $V = 100$ litre, are connected together with pipes as shown in the figure below.



The tanks are filled with salt water; $x_1(t)$ and $x_2(t)$ are the mass in grammes of salt in the respective tanks at time t. Salt water flows from tank T_1 to tank T_2, and equally from T_2 to T_1, at the rate $q = 1$ litres per second in each direction. We ignore the volume of the pipes, and assume instantaneous mixing of salt water [. . .] Let $x_1(0) = 100g$ and $x_2(0) = 0g$. At what time is $x_2(t) = 25g$?

Observation

We need to find two unknown (differentiable) functions. Varying time from t to $t + \Delta t$ we see

$$
\begin{aligned}
x_1(t + \Delta t) &= x_1(t) - \frac{\Delta t}{100} x_1(t) + \frac{\Delta t}{100} x_2(t) \\
x_2(t + \Delta t) &= x_2(t) + \frac{\Delta t}{100} x_1(t) - \frac{\Delta t}{100} x_2(t)
\end{aligned}
$$

For $\Delta t \to 0$ we see

$$
\begin{aligned}
x_1'(t) &= -\frac{1}{100} x_1(t) + \frac{1}{100} x_2(t) \\
x_2'(t) &= \frac{1}{100} x_1(t) - \frac{1}{100} x_2(t)
\end{aligned}
$$

This combines linear systems and linear differential equations!

19.1 Definition

A **system of linear differential equations** is given by

$$\begin{aligned} x_1' &= a_{11}x_1 + \cdots + a_{1n}x_n \\ x_2' &= a_{21}x_2 + \cdots + a_{2n}x_n \\ \vdots & \qquad\quad \vdots \\ x_n' &= a_{n1}x_1 + \cdots + a_{nn}x_n \end{aligned}$$

Here $x_1, \ldots, x_n$ are unknown differentiable functions of t with derivatives $x_1', \ldots, x_n'$ and the a_{ij} are linear.

A **solution** to the system is a family of differentiable functions $x_1, \ldots, x_n$ such that the equations are simultaneously true.

Note: We have exactly as many unknown functions as there are equations!

Rewriting the system as a matrix equation

We can construct vectors from differentiable functions and define a derivative component wise:

$$\overrightarrow{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad \overrightarrow{x}'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}$$

Addition and scalar multiplication for these vectors of functions works again component wise.
For a matrix $A = \begin{bmatrix} \overrightarrow{a}_1 \cdots \overrightarrow{a}_n \end{bmatrix}$ define as usual

$$A\overrightarrow{x}(t) = \overrightarrow{a}_1 x_1(t) + \cdots \overrightarrow{a}_n x_n(t)$$

With $A = \left[a_{ij}\right]_{1 \le i,j \le n}$ the system of linear differential equations can be rewritten as a matrix equation:

$$\overrightarrow{x}'(t) = A\overrightarrow{x}(t)$$

Example: The system from the exam reads in matrix form

$$\overrightarrow{x}'(t) = \frac{1}{100}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\overrightarrow{x}(t)$$

To find a strategy to solve such systems we consider first the special case when the matrix A is a diagonal matrix.

Strategy to solve systems of linear differential equations

Consider the system of linear differential equations $\overrightarrow{x}'(t) = A\overrightarrow{x}$.

Note: A must be diagonalizable for this method to work!

1. Compute the eigenvalues $\{ \lambda_1, \ldots, \lambda_k \}$ of A together with a basis of eigenvectors $\{ \overrightarrow{v}_1, \ldots, \overrightarrow{v}_k \}$.
2. For each $\overrightarrow{v}_r$ let λ_r be the associated eigenvalue. The general solution is then

$$\overrightarrow{x}(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t} \overrightarrow{v}_i \tag{1}$$

3. If there are initial conditions, choose the c_i in the general solution (1) such that

$$\overrightarrow{x}(0) = \sum_{i=1}^{n} c_i \overrightarrow{v}_i.$$

   Then (1) with these coefficients solves the problem with given initial conditions.

Strategy to solve systems of linear differential equations II

If we are given a real matrix which is only diagonalisable over $\mathbb{C}$, we have to modify (1):

4. For each complex eigenvalue $\lambda = a + ib$ with complex eigenvector $\overrightarrow{v}_\lambda = \overrightarrow{v}_r + i\overrightarrow{v}_i$ the corresponding part of the general solution is:

   $$(r(\cos(bt)\overrightarrow{v}_r - \sin(bt)\overrightarrow{v}_i) + s(\cos(bt)\overrightarrow{v}_i + \sin(bt)\overrightarrow{v}_r))e^{at}$$

   for $r, s \in \mathbb{R}$.
   Thhen we remove the part of the general solution corresponding to the conjugate eigenvalue $\overline{\lambda}$

Eigenvalues govern the behavior of solutions

Its not always practical to solve systems of differential equations in the above way. However, the eigenvalues govern the behaviour of the system:

- The eigenvalue λ with the largest absolute value dominates the behaviour. For almost all initial conditions, $\overrightarrow{x}(t) \to e^{\lambda t}\overrightarrow{v}_\lambda$
- If all eigenvalues are positive, $\overrightarrow{x}(t)$ grows exponentially (except for $\overrightarrow{0}$). The origin is called a **source**.
- If all eigenvalues are negative, $\overrightarrow{x}(t) \to \overrightarrow{0}$ exponentially. The origin is called a **sink**.
- If some are positive and some negative, then for some starting conditions $\overrightarrow{x}(t) \to \overrightarrow{0}$ but for the rest $\overrightarrow{x}(t)$ will grow exponentially. The origin is called a **saddle**.
- If the eigenvalues are complex, previous points apply to their real part. The imaginary part adds in a rotation component.