TMA 4115 Matematikk 3

Lecture 24 for MBIOT5, MTKJ, MTNANO

Alexander Schmeding

NTNU

26. March 2014

19.1 Definition

A **system of linear differential equations** is given by

$$\begin{aligned} x_1' &= a_{11}x_1 + \cdots + a_{1n}x_n \\ x_2' &= a_{21}x_2 + \cdots + a_{2n}x_n \\ \vdots & \qquad\quad \vdots \\ x_n' &= a_{n1}x_1 + \cdots + a_{nn}x_n \end{aligned}$$

Here $x_1, \ldots, x_n$ are unknown differentiable functions of t with derivatives $x_1', \ldots, x_n'$ and the a_{ij} are linear.

A **solution** to the system is a family of differentiable functions $x_1, \ldots, x_n$ such that the equations are simultaneously true.

With $A = \left[a_{ij}\right]_{1 \leq i,j \leq n}$ the system of linear differential equations can be rewritten as a matrix equation:

$$\overrightarrow{x}'(t) = A\overrightarrow{x}(t)$$

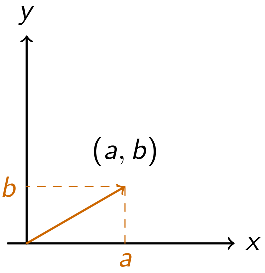
Strategy to solve systems of linear differential equations

We can solve the system of linear differential equations $\overrightarrow{x}'(t) = A\overrightarrow{x}$ **if** A is diagonalizable. General strategy:

1. Compute eigenvalues $\{\lambda_1, \ldots, \lambda_k\}$ with eigenvectors $\{\overrightarrow{v}_1, \ldots, \overrightarrow{v}_k\}$ of A to find a general solution
$$\overrightarrow{x}(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t} \overrightarrow{v}_i \tag{1}$$
2. If an eigenvalues $\lambda = a + ib$ is complex, the general solution is complex and we have to construct a real valued solution. Omit the part corresponding to the complex conjugate $\overline{\lambda}$ and replace the λ part in (1) by
$$(r(\cos(bt)\overrightarrow{v}_r - \sin(bt)\overrightarrow{v}_i) + s(\cos(bt)\overrightarrow{v}_i + \sin(bt)\overrightarrow{v}_r))e^{at},$$
for $r, s \in \mathbb{R}$
3. With initial conditions, choose c_i in (1) such that
$$\overrightarrow{x}(0) = \sum_{i=1}^{n} c_i \overrightarrow{v}_i.$$

Geometry of $\mathbb{R}^2$: Length of vectors

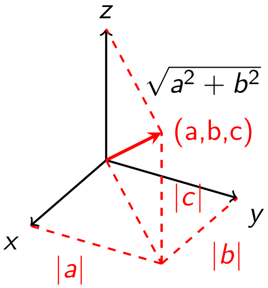
Length in 2d:



Pythagoras: length of $\begin{bmatrix} a \\ b \end{bmatrix}$

is $\sqrt{a^2 + b^2}$

Length in 3d:

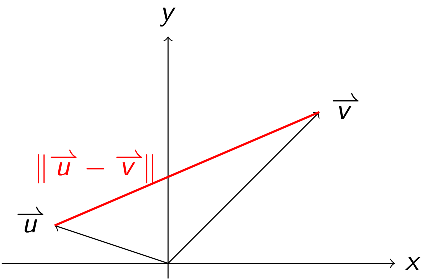


Pythagoras: length of $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

is $\sqrt{a^2 + b^2 + c^2}$

Distance between two vectors

The length of vectors allows us to measure the distance between points:

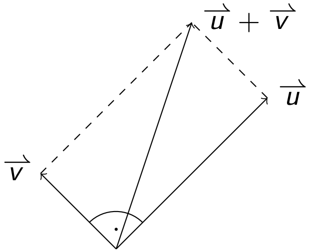


We view $\|\overrightarrow{u}-\overrightarrow{v}\|$ as the distance between $\overrightarrow{u}$ and $\overrightarrow{v}$ and set $\mathrm{dist}(\overrightarrow{u},\overrightarrow{v})=\|\overrightarrow{u}-\overrightarrow{v}\|$.

Note: $\|\overrightarrow{u}-\overrightarrow{v}\|=\|\overrightarrow{v}-\overrightarrow{u}\|$.

Pythagoras Theorem in 2d

If the two vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ in $\mathbb{R}^2$ meet in a right angle, we say they are **perpendicular** (or **orthogonal**) to each other:



Pythagoras theorem then states:

$$\|\overrightarrow{u} + \overrightarrow{v}\|^2 = \|\overrightarrow{u}\|^2 + \|\overrightarrow{v}\|^2 \tag{2}$$

Comparing both sides of (2), the equation holds if and only if:

$$\overrightarrow{u} \cdot \overrightarrow{v} = 0$$

Idea: Use this to define orthogonal vectors in general settings.