# TMA 4115 Matematikk 3

## Lecture 25 for MBIOT5, MTKJ, MTNANO

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# Inner product, length and orthogonality

Let $\overrightarrow{x}, \overrightarrow{y}$ be vectors in $\mathbb{R}^n$ with components $x_i$ and $y_i$, respectively.

**Dot product/ Inner product** $\overrightarrow{x} \cdot \overrightarrow{y} = \overrightarrow{x}^T \overrightarrow{y} = \sum_{i=1}^{n} x_i y_i$.

Can use this to define

- **Length** of a vector: $\|\overrightarrow{x}\| = \sqrt{\overrightarrow{x} \cdot \overrightarrow{x}}$,
- **Distance** between $\overrightarrow{x}$ and $\overrightarrow{y}$: $\text{dist}(\overrightarrow{u}, \overrightarrow{v}) = \|\overrightarrow{u} - \overrightarrow{v}\|$,
- **Orthogonality** of vectors: $\overrightarrow{x}$ and $\overrightarrow{y}$ are orthogonal to each other if and only if $\overrightarrow{x} \cdot \overrightarrow{y} = 0$.

# Orthogonal Complement

Let $W \subseteq \mathbb{R}^n$ be non-empty. We say

- $\overrightarrow{z} \in \mathbb{R}^n$ is orthogonal to $W$ if $\forall \overrightarrow{v} \in W$ we have $\overrightarrow{v} \cdot \overrightarrow{z} = 0$
- The set $W^\perp$ of all vectors in $\mathbb{R}^n$ orthogonal to $W$ is called the **orthogonal complement** of $W$.

**Example**: $\{\overrightarrow{0}\}^\perp = \mathbb{R}^n$ and $(\mathbb{R}^n)^\perp = \{\overrightarrow{0}\}$.

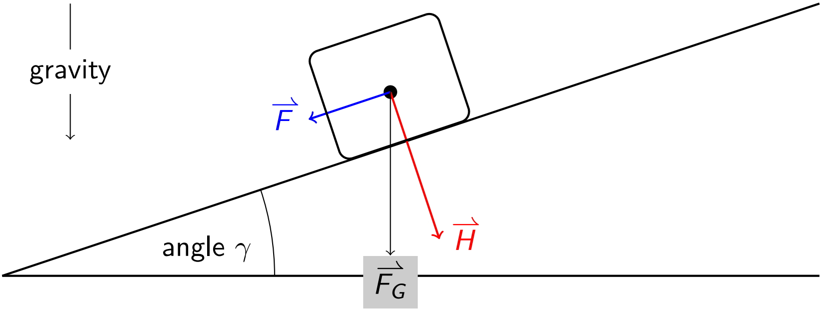
$$L = \text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\} \subseteq \mathbb{R}^2 \text{ then } L^\perp = \text{span}\left\{\begin{bmatrix}-1\\1\end{bmatrix}\right\}.$$

$$\text{furthermore } \left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}^\perp = \text{span}\left\{\begin{bmatrix}-1\\1\end{bmatrix}\right\} = L^\perp$$

$$\text{If } P \text{ is the } x-y\text{-plane in } \mathbb{R}^3 \text{ then } P^\perp = \text{span}\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$$

# Splitting forces in physics

Want to split a force into independent components. Say we know the weight of a block on a slope:




Can compute $\overrightarrow{F}_G$ from the weight but we want: $\overrightarrow{F}$, the force acting on the block in the direction of the slope.

**Note:** $\overrightarrow{F}$ and $\overrightarrow{H}$ are orthogonal!

$\rightarrow$ Idea: split $\overrightarrow{F}_G$ in orthogonal components

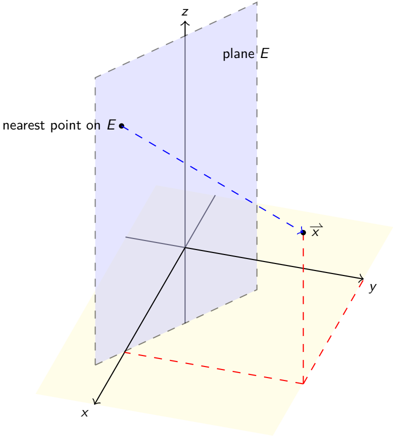
# Nearest points on subspaces

We can view the result of the orthogonal projection as a nearest point on a given subspace in $\mathbb{R}^n$.

Consider a vector $\overrightarrow{x}$ in $\mathbb{R}^3$

Nearest point $\overrightarrow{v}$ on plane: of all plane points, $\overrightarrow{v}$ minimizes distance to $\overrightarrow{x}$

**Observe**: $\overrightarrow{x} - \overrightarrow{v}$ is perpendicular to $E$ (i.e. it is in $E^{\perp}$)



## 21.7 The Gram-Schmidt Process

Let $\{\overrightarrow{x}_1, \ldots, \overrightarrow{x}_p\}$ be a basis for a non-zero subspace $W \subseteq \mathbb{R}^n$.Define

$$
\begin{aligned}
\overrightarrow{v}_1 &= \overrightarrow{x}_1 \\
\overrightarrow{v}_2 &= \overrightarrow{x}_2 - \frac{\overrightarrow{x}_2 \cdot \overrightarrow{v}_1}{\overrightarrow{v}_1 \cdot \overrightarrow{v}_1}\overrightarrow{v}_1 \\
\overrightarrow{v}_3 &= \overrightarrow{x}_3 - \frac{\overrightarrow{x}_3 \cdot \overrightarrow{v}_1}{\overrightarrow{v}_1 \cdot \overrightarrow{v}_1}\overrightarrow{v}_1 - \frac{\overrightarrow{x}_3 \cdot \overrightarrow{v}_2}{\overrightarrow{v}_2 \cdot \overrightarrow{v}_2}\overrightarrow{v}_2 \\
\vdots &= \vdots \qquad \vdots \\
\overrightarrow{v}_p &= \overrightarrow{x}_p - \sum_{i=1}^{p-1} \frac{\overrightarrow{x}_p \cdot \overrightarrow{v}_i}{\overrightarrow{v}_i \cdot \overrightarrow{v}_i}\overrightarrow{v}_i
\end{aligned}
$$

Then $\{\overrightarrow{v}_1, \ldots, \overrightarrow{v}_p\}$ is an orthogonal basis for $W$ and in addition

$$\text{span}\,\{\overrightarrow{v}_1, \ldots, \overrightarrow{v}_k\} = \text{span}\,\{\overrightarrow{x}_1, \ldots, \overrightarrow{x}_k\} \quad \text{for } 1 \leq k \leq p$$