TMA 4115 Matematikk 3

Lecture 26 for MBIOT5, MTKJ, MTNANO

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Inner product, length and orthogonality

Let $\overrightarrow{x}, \overrightarrow{y}$ be vectors in $\mathbb{R}^n$ with components x_i and y_i, respectively.

Dot product/ Inner product $\overrightarrow{x} \cdot \overrightarrow{y} = \overrightarrow{x}^T \overrightarrow{y} = \sum_{i=1}^n x_i y_i$.

Define **Orthogonality** of vectors: $\overrightarrow{x}$ and $\overrightarrow{y}$ are orthogonal to each other if and only if $\overrightarrow{x} \cdot \overrightarrow{y} = 0$.

For a subspace $W \subseteq \mathbb{R}^n$ we can split $\overrightarrow{y} \in \mathbb{R}^n$ into

$$\overrightarrow{y} = \overrightarrow{y}_W + \overrightarrow{z}, \quad \overrightarrow{y}_W \in W, \ \overrightarrow{z} \in W^\perp$$

and obtain an orthogonal projection $\text{proj}_W \colon \mathbb{R}^n \to W, \overrightarrow{y} \mapsto \overrightarrow{y}_W$.
If $\{ \overrightarrow{v}_1, \ldots, \overrightarrow{v}_p \}$ is an orthogonal basis for W, its easy to compute

$$\overrightarrow{y}_W = \sum_{i=1}^{p} \frac{\overrightarrow{y} \cdot \overrightarrow{v}_i}{\overrightarrow{v}_i \cdot \overrightarrow{v}_i} \overrightarrow{v}_i$$

21.7 The Gram-Schmidt Process

Let $\{\overrightarrow{x}_1, \ldots, \overrightarrow{x}_p\}$ be a basis for a non-zero subspace $W \subseteq \mathbb{R}^n$. Define

$$\begin{aligned}
\overrightarrow{v}_1 &= \overrightarrow{x}_1 \\
\overrightarrow{v}_2 &= \overrightarrow{x}_2 - \frac{\overrightarrow{x}_2 \cdot \overrightarrow{v}_1}{\overrightarrow{v}_1 \cdot \overrightarrow{v}_1}\overrightarrow{v}_1 \\
\overrightarrow{v}_3 &= \overrightarrow{x}_3 - \frac{\overrightarrow{x}_3 \cdot \overrightarrow{v}_1}{\overrightarrow{v}_1 \cdot \overrightarrow{v}_1}\overrightarrow{v}_1 - \frac{\overrightarrow{x}_3 \cdot \overrightarrow{v}_2}{\overrightarrow{v}_2 \cdot \overrightarrow{v}_2}\overrightarrow{v}_2 \\
\vdots &= \vdots \qquad \vdots \\
\overrightarrow{v}_p &= \overrightarrow{x}_p - \sum_{i=1}^{p-1} \frac{\overrightarrow{x}_p \cdot \overrightarrow{v}_i}{\overrightarrow{v}_i \cdot \overrightarrow{v}_i}\overrightarrow{v}_i
\end{aligned}$$

Then $\{\overrightarrow{v}_1, \ldots, \overrightarrow{v}_p\}$ is an orthogonal basis for W and in addition

$$\text{span}\,\{\overrightarrow{v}_1, \ldots, \overrightarrow{v}_k\} = \text{span}\,\{\overrightarrow{x}_1, \ldots, \overrightarrow{x}_k\} \quad \text{for } 1 \leq k \leq p$$

Application for Gram-Schmidt

(Spring 2011 Problem 5)

(a) Find an orthogonal basis of $\mathrm{Col}(A)$ with

$$A = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 1 & 5 & -3 \\ -1 & -1 & -2 & 1 \end{bmatrix}$$

(b) Find for $\overrightarrow{y} = \begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix}$ the nearest point in $\mathrm{Col}(A)$

To run Gram-Schmidt we need a basis of $\mathrm{Col}(A)$
($\rightarrow$ Gauss elimination on A!)

Alternatively, we can apply Gram-Schmidt to a generating system of $\mathrm{Col}(A)$. This is less work!
Let $\overrightarrow{x}_i$ be the i-th column of A.

Gram Schmidt on a generating system

We note that all vectors in the generating set are non-zero. Thus set $\overrightarrow{v}_1 = \overrightarrow{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ then $\overrightarrow{v}_2 = \overrightarrow{x}_2 - \frac{\overrightarrow{x}_2 \cdot \overrightarrow{v}_1}{\overrightarrow{v}_1 \cdot \overrightarrow{v}_1} \overrightarrow{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$,

$\overrightarrow{v}_3 = \overrightarrow{x}_3 - \frac{\overrightarrow{x}_3 \cdot \overrightarrow{v}_1}{\overrightarrow{v}_1 \cdot \overrightarrow{v}_1} \overrightarrow{v}_1 - \frac{\overrightarrow{x}_3 \cdot \overrightarrow{v}_2}{\overrightarrow{v}_2 \cdot \overrightarrow{v}_2} \overrightarrow{v}_2 = \overrightarrow{0}$

we get $\overrightarrow{0}$, so $\overrightarrow{x}_3$ was already contained in the span of $\overrightarrow{v}_1, \overrightarrow{v}_2$

$\rightsquigarrow$ discard $\overrightarrow{v}_3$ and continue with the next vector $\overrightarrow{x}_4$.

$\overrightarrow{v}_3 = \overrightarrow{x}_4 - \frac{\overrightarrow{x}_4 \cdot \overrightarrow{v}_1}{\overrightarrow{v}_1 \cdot \overrightarrow{v}_1} \overrightarrow{v}_1 - \frac{\overrightarrow{x}_4 \cdot \overrightarrow{v}_2}{\overrightarrow{v}_2 \cdot \overrightarrow{v}_2} \overrightarrow{v}_2 = \overrightarrow{0}$

again the result ist $\overrightarrow{0}$ and we discard $\overrightarrow{x}_4$.Since it was the last, the orthogonal basis of Col(A) is $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$

Application of Gram-Schmidt

To answer part (b), we recall the

21.6 Best approximation theorem Let $W \subseteq \mathbb{R}^n$ be a subspace and $\overrightarrow{y} \in \mathbb{R}^n$. Then $\overrightarrow{y}_W = \text{proj}_W(\overrightarrow{y})$ is closest in W to $\overrightarrow{y}$:

$$\|\overrightarrow{y} - \overrightarrow{y}_W\| < \|\overrightarrow{y} - \overrightarrow{v}\|, \quad \overrightarrow{v} \in W \setminus \{\overrightarrow{y}_W\}$$

We need to compute $\overrightarrow{y}_{\text{Col}(A)}$ as the nearest point to $\overrightarrow{y}$ in $\text{Col}(A)$.

$$\overrightarrow{y}_{\text{Col}(A)} = \frac{\overrightarrow{y} \cdot \overrightarrow{v}_1}{\overrightarrow{v}_1 \cdot \overrightarrow{v}_1}\overrightarrow{v}_1 + \frac{\overrightarrow{y} \cdot \overrightarrow{v}_2}{\overrightarrow{v}_2 \cdot \overrightarrow{v}_2}\overrightarrow{v}_2 = \begin{bmatrix} 0 \\ 5 \\ -2 \end{bmatrix}$$

Weight of baggage

We want to know the weight x of our suitcase.
If it is too heavy we have to pay extra on the airplane and we don't want that.

On a scale

weight $x = x_1$kg

Second scale

weight $x = x_2$kg

Third scale

weight $x = x_3$kg

What is the weight of our suitcase?

Weight of baggage II

From our scales we have the information:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leftrightarrow \begin{cases} x & = x_1 \\ x & = x_2 \\ x & = x_3 \end{cases}$$

Unless $x_1 = x_2 = x_3$ holds (by chance), the linear system is **inconsistent**, i.e. there is no exact solution.

Problem: We want an "approximate" solution. This means a number y such that the errors $|x_i - y|$ are as small as possible.

Weight of baggage III

Idea: If $A\overrightarrow{x} = \overrightarrow{b}$ is inconsistent, approximate solution means an $\overrightarrow{x}$ such that $A\overrightarrow{x}$ is as near as possible to the target vector $\overrightarrow{b}$.

$A\overrightarrow{x}$ minimizes the distance to $\overrightarrow{b}$ if it is $\text{proj}_{\text{Col}(A)}(\overrightarrow{b})$!
This is in the column space, and we can then solve $A\overrightarrow{x} = \text{proj}_{\text{Col}(A)}(\overrightarrow{b})$.

In our example with $A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\overrightarrow{b} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, we set $\overrightarrow{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and obtain

$$\text{proj}_{\text{Col}(A)}(\overrightarrow{b}) = \frac{\overrightarrow{b} \cdot \overrightarrow{v}_1}{\overrightarrow{v}_1 \cdot \overrightarrow{v}_1} \overrightarrow{v_1} = \begin{bmatrix} \frac{x_1+x_2+x_3}{3} \\ \frac{x_1+x_2+x_3}{3} \\ \frac{x_1+x_2+x_3}{3} \end{bmatrix}$$

Thus $Ax = \text{proj}_{\text{Col}(A)}(\overrightarrow{b})$ shows: $x = \frac{x_1+x_2+x_3}{3}$.