TMA 4115 Matematikk 3

Lecture 4 for MBIOT5, MTKJ, MTNANO

Alexander Schmeding

NTNU

15. January 2014

We already know how to solve the differential equations

$$y'(t) = \frac{d}{dt}y(t) = f(y,t)$$

if the function f is "nice".

We call such an equation a *first order differential equation*, because it **only involves the first derivative** of the unknown y.

First order differential equations are "simple". The (physical) world is complicated, hence (in general) more complicated differential equations describe the real world.

3.1 Newtons second law

The *acceleration* a of a body with *mass* m is proportional to the *net force* F via

$$F = ma \tag{1}$$

Question: What is the displacement $y(t)$ of the body from a reference point?

Notice:

Acceleration a is rate of change of velocity v, i.e. $a = \frac{d}{dt}v = v'$

Velocity v is rate of change of the displacement, i.e. $v = \frac{d}{dt}y = y'$

The net force F usually depends on time t, velocity and displacement, i.e. $F = F(t, y, v) = F(t, y, y')$

We can thus rewrite (1) as

$$F(t, y, y') = my''$$

We call this a second order differential equation since it involves derivatives of y of up to second order.

3.2 Definition

A *second order differential equation* is an equation of the form

$$\frac{d^2}{dt^2}y(t) = f(y, \frac{d}{dt}y, t) \tag{2}$$

where f is given function.

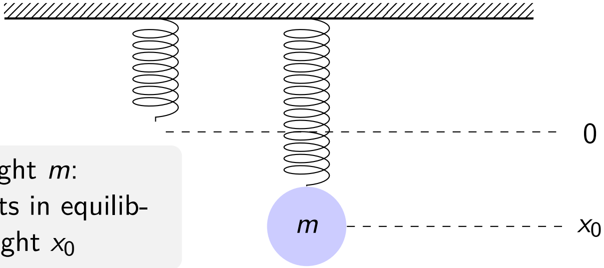
A *solution* to the second order equation (2) is a function y which is twice continuously differentiable and satisfies (2).

Usually we write $y' = \frac{d}{dt}y(t)$ and $y'' = \frac{d^2}{dt^2}y(t)$, thus (2) reads:

$$y'' = f(t, y, y') \tag{3}$$

3.3 The vibrating spring

We consider a spring suspended from a beam:



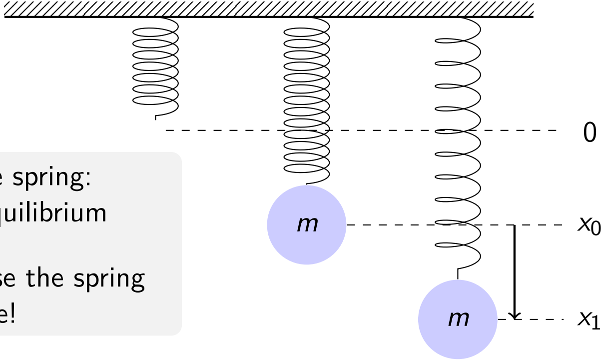
Attach weight m:
System rests in equilibrium at height x_0

Forces acting on weight: **gravity** mg and **restoring force** $R(x)$ depending on the stretch distance x.

In equilibrium, the spring does not move.

3.3 The vibrating spring

We consider a spring suspended from a beam:



Stretch the spring:
it leaves equilibrium

If we release the spring
it will move!

Forces acting on weight in motion: **damping force** $D(v)$ depending on velocity v, **external force** $F(t)$ and $R(x)$, mg.

A model for the displacement x of the spring

Recall: velocity $v = x'$ and acceleration $a = v' = x''$.

Then second Newtons law (1) yields

$$\begin{aligned} ma &= \text{total force acting on the weight} \\ &= R(x) + mg + D(v) + F(t) \end{aligned} \tag{4}$$

We rewrite (4) as

$$mx'' = R(x) + mg + D(x') + F(t). \tag{5}$$

For some springs *Hooke's law* states $R(x) = -kx$ for $k > 0$ constant and small x.
Assuming Hooke's law, (4) becomes

$$mx'' = -kx + mg + D(x') + F(t). \tag{6}$$

Is there a solution for every 2nd order differential equation?

If the equation is "nice enough" there is a solution:

3.4 Theorem
Let $p(t), q(t)$ and $g(t)$ be functions which are continuous on the interval (α, β). Fix $t_0 \in (\alpha, \beta)$ and $y_0, y_1 \in \mathbb{R}$. There is one and only one function $y\colon (\alpha, \beta) \to \mathbb{R}$ which solves

$$\begin{cases} y'' + p(t)y' + q(t)y = g(t), & \text{for } t \in (\alpha, \beta) \\ y(t_0) = y_0, y'(t_0) = y_1 \end{cases} \tag{7}$$

3.5 Remark: The solution of (7) is defined on all of (α, β)
We need $y(t_0) = y_0, y'(t_0) = y_1$ to get a unique solution.

Open Problem: How to find a solution?