TMA 4115 Matematikk 3

Lecture 5 for MBIOT5, MTKJ, MTNANO

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The **order** of a differential equation is the highest non-trivial derivative appearing.

We study linear (second order) differential equations:

$$y'' + p(t)y' + q(t)y = g(t)$$

where p, q and g are given functions.

If the forcing term g is 0, we call the equation **homogeneous**, otherwise **inhomogeneous**.

An **initial value problem** (IVP) is a differential equation with enough initial values to specify a solution.

Theorem If p, q and g are continuous functions defined on (α, β) and $t_0 \in (\alpha, \beta)$ the IVP

$$\begin{cases} y'' + p(t)y' + q(t)y = g(t) & t \in (\alpha, \beta) \\ y(t_0) = y_0, y'(t_0) = y_1 \end{cases}$$

has a unique solution.

Two functions $u, v \colon (\alpha, \beta) \to \mathbb{R}$ are **linearly independent** (on (α, β)) if there is no $C \in \mathbb{R}$ with $u(t) = Cv(t)$ for all $t \in (\alpha, \beta)$.

Test linear independence: **Inspection** or
if u, v solve the same linear homogeneous differential equation use their **Wronskian** $W(t) = u(t)v'(t) - v(t)u'(t)$.

Structure of the general solution

3.18 Theorem Let y_1, y_2 be linearly independent solutions to

$$y'' + p(t)y' + q(t)y = 0$$

Then the **general solution** to the differential equation is

$$y(t) = Ay_1(t) + By_2(t)$$

where $A, B \in \mathbb{R}$.

We call two linearly independent solutions for a second order homogeneous linear equation a **fundamental set** of solutions.

Strategy to solve homogeneous linear differential equations

$$y'' + p(t)y' + q(t)y = 0$$

- Find two linearly independent solutions u, v to the equation
- Check linear independence by inspection or by showing that their Wronskian $W(t)$ is non-zero
- Obtain general solution $Au + Bv$

To *solve the IVP* $y'' + p(t)y' + q(t)y = 0, y(t_0) = y_0, y'(t_0) = y_1$:

- Construct a general solution to $y'' + p(t)y' + q(t)y = 0$
- Use $y(t_0) = y_0$ and $y'(t_0) = y_1$ to determine A and B

3.19 Example

We know that for $\omega \neq 0$, $\sin(\omega t)$ and $\cos(\omega t)$ solve

$$y'' + \omega^2 y = 0 \tag{11}$$

Their Wronskian is $W(t) = \omega(\cos(\omega t)^2 + \sin(\omega t)^2) = \omega \neq 0$. Hence $\{\sin(\omega t), \cos(\omega t)\}$ is a fundamental set of solutions and

$$y(t) = A\sin(\omega t) + B\cos(\omega t) \; A, B \in \mathbb{R} \text{ (general solution)}$$

To find the solution with initial condition $y(0) = 2$ and $y'(0) = 1$ insert the values in the general solution:

$$\begin{aligned} 2 &= y(0) = A\sin(0) + B\cos(0) = B \\ 1 &= y'(0) = A\omega\cos(0) - B\omega\sin(0) = A\omega \end{aligned}$$

Hence $y(t) = \frac{1}{\omega}\sin(\omega t) + 2\sin(\omega t)$ solves (11) with $y(0) = 2$ and $y'(0) = 1$ (IVP).

Problem: How to find any solution?

We know what to do if we already found solutions to a linear homogeneous equation. However, how do we find these solutions?

Goal: Construct solutions for homogeneous linear equations with constant coefficients.

Idea: Consider

$$y' + qy = 0, \quad q \in \mathbb{R}$$

We know that $y(t) = Ce^{-qt}$ solves the equation for all $C \in \mathbb{R}$. Try $y(t)$ as a solution to a second order homogeneous equation.