TMA 4115 Matematikk 3

Lecture 6 for MBIOT5, MTKJ, MTNANO

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A linear (second order) differential equation

$$y'' + py' + qy = g(t) \tag{1}$$

where $p, q \in \mathbb{R}$ and g is a given function is called linear (inhomogeneous) differential equation with constant coefficients.

Solve the homogeneous equation associated to (1):

$$y'' + py' + qy = 0 \tag{4}$$

using the associated **characteristic polynomial**

$$\lambda^2 + p\lambda + q = 0. \tag{5}$$

We distinguish three cases depending on the **characteristic roots**:

Solutions for equations with constant coefficients

Case 1 $p^2 - 4q > 0$, i.e. two distinct, real roots λ_1 and λ_2.
Fundamental set of solutions:

$$y_1(t) = e^{\lambda_1 t} \quad y_2(t) = e^{\lambda_2 t}$$

Case 2 $p^2 - 4q < 0$, we have two distinct, complex roots $\lambda_1 = a + ib$ and $\overline{\lambda_1}$. Fundamental set of (real valued) solutions:

$$y_1(t) = e^{at}\cos(bt) \quad y_2(t) = e^{at}\sin(bt)$$

Case 3 $p^2 - 4q = 0$, there is one repeated real root λ_1.
Fundamental set of solutions:

$$y_1(t) = e^{\lambda_1 t}, \quad y_2(t) = te^{\lambda_1 t}$$

Harmonic motion

An important example of a linear equations with constant coefficients is the equation of **harmonic motion**

$$y'' + 2cy' + \omega_0^2 y = f(t) \tag{11}$$

with $c, \omega_0 \in \mathbb{R}$ and $f(t)$ a given function.
The parameters have special names (inspired by physical meaning):

- c, the dampening parameter
- ω_0, the natural frequency
- $f(t)$, the forcing term

Example: The spring equation (Chapter 3 (6)):

$$mx'' = -kx + mg + D(x') + F(t)$$

with $D(x') = -\mu x'$ for $\mu \in [0, \infty)$ we obtain

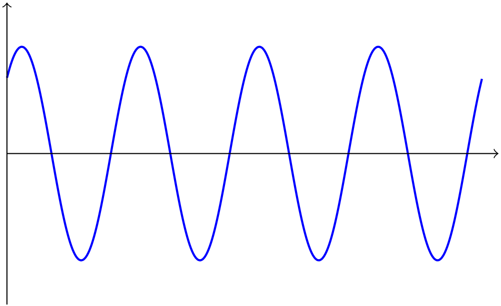
$$x'' + \frac{-\mu}{m}x' + \frac{k}{m}x = \frac{F(t)}{m} + g \tag{12}$$

Simple Harmonic motion

For $\mu = 0 = F(t) + mg$ we know a solution of (12):

$$y(t) = \cos(\omega_0 t) + \sin(\omega_0 t)$$

The solution is periodic with period $\omega_0 = \sqrt{\frac{k}{m}}$.



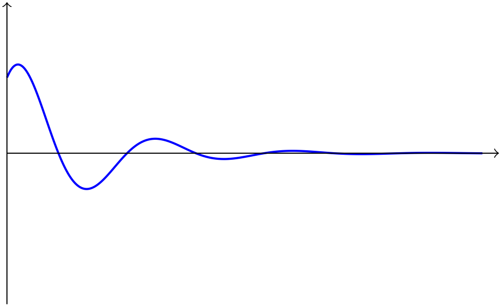
No Damping: Solution oscillates with natural frequency (here $\omega_0 = 4$)

Harmonic motion with damping $\mu \neq 0$

Set $c = \frac{-\mu}{2m}$. Several cases (see book), we only discuss $c < \omega_0$ with $c = 1, \omega_0 = 4$. Solving (12) yields

$$y(t) = e^{-ct}(a\cos(\sqrt{\omega_0^2 - c^2}t) + b\sin(\sqrt{\omega_0^2 - c^2}t))$$

We choose $a = b = 1$ and plot the solution:



Damping present: Oscillation decreases with time.

5. Inhomogeneous equations: The method of undetermined coefficients

We want to solve

$$y'' + py' + qy = f(t) \tag{1}$$

5.1 Theorem Let $y_p(t)$ be a particular solution to the inhomogeneous problem (1) and $y_1(t), y_2(t)$ be a fundamental system of solutions for the associated homogeneous equation

$$y'' + py' + qy = 0.$$

Then the general solution to the inhomogeneous equation (1) is

$$y(t) = y_p(t) + Ay_1(t) + By_2(t) \quad A, B \in \mathbb{R}.$$

Strategy: First solve the homogeneous equation associated to (1), then construct a particular solution to (1)

Problem: How to find a particular solution?

We now introduce the **method of undetermined coefficients**.

Idea: Guess a particular solution based on the forcing term.
Incorporate a parameter to stay flexible.

Guideline for the method

If the form of the forcing term f replicates under differentiation, look for a solution with the same form.

Overwiew: The method of undetermined coefficients

Forcing term $f(t)$	Trial solution	Comment
e^{rt}	ae^{rt}	
$\cos(\omega t)$ or $\sin(\omega t)$	$a\cos(\omega t) + b\sin(\omega t)$	
$P(t)$ Polynomial	$p(t)$ Polynomial	$P(t)$ and $p(t)$ have same degree, i.e.
Example: t^2	$at^2 + bt + c$	

Problem: If the trial solution is a solution of the homogeneous equation the above method does not work!

Solution: Try multiplying the trial solution with t, if that does not work multiply by t again.