TMA 4115 Matematikk 3

Lecture 7 for MBIOT5, MTKJ, MTNANO

Alexander Schmeding

NTNU

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We want to find a general solution to the inhomogeneous equation

$$y'' + py' + qy = f(t) \tag{1}$$

1. Construct a fundamental set of solutions y_1, y_2 for the homogeneous equation $y'' + py' + qy = 0$,
2. Find a particular solution $y_p(t)$ for (1),
3. The general solution is of the form $y(t) = y_p(t) + Ay_1(t) + By_2(t), A, B \in \mathbb{R}$,
4. To solve the (IVP): (1) and $y(t_0) = y_0$, $y'(t_0) = y_1$: Insert numbers in general solution to determine A and B.

If the forcing term $f(t)$ is "nice" we can use the *method of undetermined coefficients* to compute y_p in step 2.

Method of undetermined coefficients

Guess a particular solution based on the forcing term.
Incorporate a parameter to stay flexible.

Guideline for the method

If the form of the forcing term f replicates under differentiation, look for a solution with the same form.

Problem: The method fails if the forcing term is not one of the functions we have trial solutions for.

6. Inhomogeneous equations: Variation of parameters

We want to construct solutions via "variation of parameters" for

$$y'' + p(t)y' + q(t)y = f(t) \tag{1}$$

Note: We do not assume constant coefficients here!

Idea: To find a particular solution for $y' + q(t)y = f(t)$ we set

$$y_p(t) = v(t)y_h(t)$$

where y_h is a solution for the associated homogeneous equation and v is an unknown function. Differentiate to determine v. Copy this idea using a fundamental set of solutions y_1, y_2.

6.2 Summary of the method

- We need a fundamental set y_1, y_2 of solutions to $y'' + p(t)y' + q(t)y = 0$.
- Define $y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$ with unknown functions v_1, v_2
- Solve

$$v_1(t) = \int \frac{-y_2(t)f(t)}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}dt$$
$$v_2(t) = \int \frac{y_1(t)f(t)}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}dt$$

  to obtain y_p.

Alternatively, if you can't remember the formula: Derive it as explained in the lecture **6.1**.

Forced harmonic motion: Underdampend case

We consider a forced harmonic motion with a periodic forcing term

$$y'' + 2cy' + \omega_0^2 y = \cos(\omega t) \tag{6}$$

with $c, \omega_0 \in \mathbb{R}$ and

- c, the dampening parameter
- ω_0, the natural frequency
- ω, the driving frequency

Consider undampened harmonic motion, i.e. $c = 0$. A fundamental set of solutions for $y'' + \omega_0^2 y = 0$ is

$$y_1(t) = \cos(\omega_0 t) \qquad y_2(t) = \sin(\omega_0 t)$$

Case 1: $\omega \neq \omega_0$

Use undetermined coefficients to obtain a particular solution to (6) for $c = 0$.
Trial solution: $y_p(t) = A\cos(\omega t) + B\sin(\omega t)$.

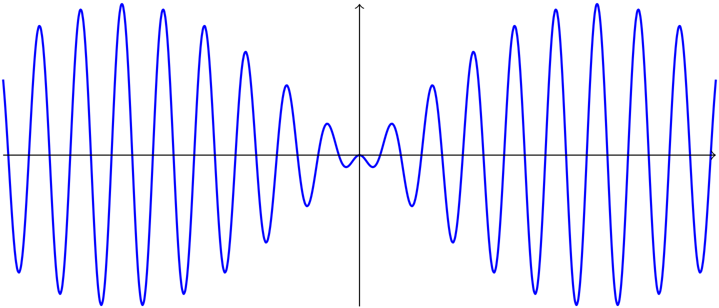
Insert in (6) and solve to find $A = \frac{1}{\omega - \omega_0}, B = 0$ whence

$$y(t) = \frac{1}{\omega - \omega_0}\cos(\omega t) + a\cos(\omega_0 t) + b\sin(\omega_0 t)$$

is the general solution to (6). With initial conditions $y(0) = 0, y'(0) = 0$ we obtain the unique solution

$$y(t) = \frac{1}{\omega - \omega_0}(\cos(\omega t) - \cos(\omega_0 t))$$

Example plot: $\omega = 12$ and $\omega_0 = 11$



Case $\omega_0 \neq \omega$: Fast oscillation with slowly varying amplitude

Case 2: $\omega = \omega_0$

For $\omega = \omega_0$ the trial solution solves the homogeneous equation. Multiply the trial solution by t and solve to find

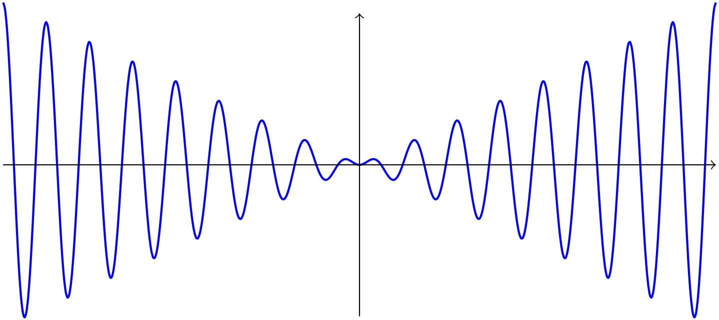
$$y(t) = \frac{1}{2\omega_0} t \sin(\omega_0 t) + a \cos(\omega_0 t) + b \sin(\omega_0 t)$$

the general solution to (6). With initial conditions $y(0) = 0, y'(0) = 0$ this reduces to

$$y(t) = \frac{1}{2\omega_0} t \sin(\omega_0 t)$$

Example plot: $\omega = \omega_0 = 11$

The solution $y(t) = \frac{t}{22}\sin(11t)$



Case $\omega_0 = \omega$: The amplitude grows linearly with time

Looking back and beyond

Our objective has been to construct solutions for

$$y'' + p(t)y' + q(t)y = f(t)$$

You might have noticed:

- Complex numbers and complex functions (exp!) were used to solve equations...
  ...even though the coefficients are real valued functions!
- Values of parameters in solutions (to solve IVP's) are determined by solving linear equations
- The Wronskian determines linear independence of solutions
- ...and in **6.1** the Wronskian showed that the system of linear equations is solvable

These observations lead to *complex numbers* and *linear algebra*.