TMA 4115 Matematikk 3

Lecture 9 for MBIOT5, MTKJ, MTNANO

Alexander Schmeding

NTNU

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Solving linear systems

Given a linear system as

$$\begin{aligned} x_1 + 5x_2 + 3x_3 + 2x_4 &= 4 \\ x_1 - 2x_3 + 2x_4 &= 0 \\ 2x_2 + 4x_3 + 2x_4 &= 1 \end{aligned} \tag{2}$$

find x_1, x_2, x_3, x_4 which simultaneously satisfy (2).

Use **elementary operations** to replace (2) with an **equivalent** system which is easier.
But first: Rewrite (2) as an **augmented matrix**

$$\begin{bmatrix} 1 & 5 & 3 & 2 & 4 \\ 1 & 0 & -2 & 2 & 0 \\ 0 & 2 & 4 & 2 & 1 \end{bmatrix}$$

Then apply elementary row operations and the Gaussian elimination to obtain a matrix in (reduced) echelon form.

Row operations & Gaussian elimination

Elementary row operations:

- Interchange two rows,
- add a multiple of another row to a row, or
- multiply by a non-zero number.

Gaussian elimination uses these operations, to compute (reduced) Echelon form.

Idea: Start in the left upper corner and work to the bottom right, using elementary operations.

The steps to compute Echelon form are usually carried out first. We call this the **forward phase** (from left upper corner down). Transforming from Echelon to reduced Echelon form is called the **backward phase** (from right bottom move up, to reduce computation).

Pivot positions

Theorem: For each matrix, there is a unique matrix in reduced echelon form such that both matrices are equivalent via elementary row operations.

Pivot positions are positions in a matrix which coincide with a leading one in reduced echelon form

$$\begin{bmatrix} 1 & 5 & 3 & 2 & 4 \\ 1 & 0 & -2 & 2 & 0 \\ 0 & 2 & 4 & 2 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 4 & -\frac{6}{10} \\ 0 & 1 & 0 & -1 & 1.1 \\ 0 & 0 & 1 & 1 & -\frac{3}{10} \end{bmatrix}$$

A **pivot column** is a column with a pivot position.
The reduced echelon matrix yields solutions for (2):
$x_1 = -\frac{6}{10} - 4x_4$, $x_2 = x_4 + 1.1$, $x_3 = -\frac{3}{10} - x_4$, $x_4 \in \mathbb{R}$.

Basic variables and free variables

7.10 Definition A variable x_i of a linear system is called ...

- **basic variable** if it corresponds to a pivot column,
- **free variable** if the associated column does not have a pivot position.

7.11 Theorem A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column.
If the linear system is consistent, then there
(a) is a unique solution if there are no free variables
(b) infinitely many solutions if there are free variables

Revisiting a chemical reaction equation

$$\text{Ethanol} + \text{Oxygen} \longrightarrow \text{Carbondioxide} + \text{Water}$$
$$C_2H_6O + O_2 \longrightarrow CO_2 + H_2O$$

we generated (somehow) the linear equations

$$\begin{aligned} 2x_1 + 0x_2 - 1x_3 - 0x_4 &= 0 \\ 6x_1 + 0x_2 - 0x_3 - 2x_4 &= 0 \\ 1x_1 + 2x_2 - 2x_3 - 1x_4 &= 0 \end{aligned} \tag{1}$$

How can we do this in a more systematic way?
Idea: The chemical formulae are just lists how many atoms of different kinds are present!

Can write: $C_2H_6O \rightsquigarrow \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix}$, $O_2 \rightsquigarrow \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$, $CO_2 \rightsquigarrow \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $H_2O \rightsquigarrow \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

We call the ordered lists obtained in this way **vectors** and naïvely write (1) as follows:

$$x_1 \cdot \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} - x_3 \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - x_4 \cdot \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (2)$$

Note: At the moment we have not defined $+, -, \cdot$ for vectors!

However, we would like $+, -, \cdot$ to be defined such that (2) means the same as equation (1)

Define a **vector** as an ordered list of numbers. Later there will be an abstract concept called **vector space** which leads to a refinement of our vector concept.