Forced oscillations with damping

February 4, 2014

Foreword

The goal of this example is to sum up several techniques we have seen. We will use complex numbers (modulus and argument of a complex number, real part, multiplying by the conjugate quantity. . .); we will use differential equation techniques (find the roots of the characteristic polynomial to solve the homogeneous equation; find a particular solution). It is important that you understand how to solve such an equation.

The last part gives motivations on why this equation is important in physics, and what it corresponds to. You can find it interesting, but it doesn't contain so many techniques that you are required to know.

The equation

The goal is to study the following differential equation

$$y'' + 2y' + 9y = \cos(\omega t).$$

It corresponds to an harmonic oscillator with damping and with an outside force being applied.

Goal 1: find the general solution of the equation

We have seen in class that it is done in two steps:

1. find the general solution of the homogeneous equation $y'' + 2y' + 9y = 0$;
2. find one particular solution of the equation $y'' + 2y' + 9y = \cos(\omega t)$.

General solution of the homogeneous equation

We know that it is enough to find *two independent solutions* (which we also called a fundamental system of solutions). To do this, solve

$$\lambda^2 + 2\lambda + 9 = 0$$

(this is the characteristic polynomial of the homogeneous equation)

We solve: $\Delta = 4 - 36 = -32$, so the solutions are $-1 \pm i2\sqrt{2}$. From here, we have two choices to express a fundamental system of solutions:

1. If we like complex-valued functions, we pick

$$z_1(t) = e^{(-1+2\sqrt{2})t}, \qquad z_2(t) = e^{(-1-2\sqrt{2})t}.$$

   It can be rewritten as:

$$z_1(t) = e^{-t}e^{2\sqrt{2}t}, \qquad z_2(t) = e^{-t}e^{-2\sqrt{2}t}$$

   or

$$z_1(t) = e^{-t}(\cos(2\sqrt{2}t)+i\sin(2\sqrt{2}t)), \quad z_2(t) = e^{-t}(\cos(2\sqrt{2}t)-i\sin(2\sqrt{2}t)).$$

   Note that $z_2 = \overline{z_1}$.

2. If we don't like complex numbers, we see that the real part of the roots is -1 and the imaginary part is $\pm 2\sqrt{2}$. So a fundamental system of solutions is:

$$y_1 = e^{-t}\cos(2\sqrt{2}t) \qquad y_2 = e^{-t}\sin(2\sqrt{2}t).$$

We found a fundamental system. Remember that the general solution of the **homogeneous** equation is of the form $y_h = C_1y_1 + C_2y_2$ for some constant terms C_1, C_2. (or $z_h = C_1z_1 + C_2z_2$ for some *complex* constant terms C_1, C_2 if we prefer to express it as a complex function).

Particular solution of the non-homogeneous equation

We will use undetermined coefficients method. We could look for a solution of the form

$$y_p = a\cos(\omega t) + b\sin(\omega t).$$

We would substitute y_p in the equation and look for which values of a and b this is an actual solution. However, between y_p, y_p' and y_p'', it would be a lot of terms. It is maybe simpler to switch to complex numbers.

The principle: $\cos(\omega t)$ is the real part of $e^{i\omega t}$. Therefore, we will look for a particular solution z_p of the equation

$$z'' + 2z' + 9z = e^{i\omega t}.$$

To get "back to the real world", we will let y_p be the real part of z_p.

So we look for $z_p = ae^{i\omega t}$, so $z_p' = ai\omega e^{i\omega t}$ and $z_p'' = -a\omega^2e^{i\omega t}$. We substitute in the equation:

$$-a\omega^2e^{i\omega t} + 2ai\omega e^{i\omega t} + 9ae^{i\omega t} = e^{i\omega t}.$$

We now want to solve this equation for a. Divide everything by $e^{i\omega t}$, and get:

$$\begin{aligned} -a\omega^2 + 2\omega ai + 9a &= 1 \\ a(9 - \omega^2 + 2\omega i) &= 1 \\ a &= \frac{1}{9 - \omega^2 + 2\omega i}. \end{aligned}$$

Remark that this fraction is well defined: if $9-\omega^2+2i\omega$ were zero, it would mean that both its real part and imaginary part were zero at the same time. It would mean $2i\omega = 0$, so $\omega = 0$. But if $\omega = 0$, the real part is $9 - 0^2 = 9 \neq 0$. So we are not dividing by zero, which is nice.

Note $H = 1/(9-\omega^2+2\omega i)$. It is called the *transfer function.* It is a function of ω, but it is sometimes rather seen as a function of $i\omega$: $H(i\omega) = 1/(9+(i\omega^2)+2\omega i)$.

The particular solution is

$$z_p = \frac{1}{9-\omega^2+2\omega i} e^{i\omega t}.$$

How do we get a real particular solution? There are two ways, one which tells more things than the other. Let's start by the "brute force" method. It is not recommended, but let's do it anyway.

We want to take the real part of z_p. As a first step, let's get rid of all the "i" on the denominator. For this, we multiply by the conjugate quantity:

$$\begin{aligned}\frac{1}{9-\omega^2+2\omega i} &= \frac{9-\omega^2-2\omega i}{(9-\omega^2+2\omega i)(9-\omega^2-2\omega i)}\\ &= \frac{9-\omega^2-2\omega i}{(9-\omega^2)^2+4\omega^2}\end{aligned}$$

So

$$\begin{aligned}z_p &= \frac{9-\omega^2-2\omega i}{(9-\omega^2)^2+4\omega^2}(\cos(\omega t)+i\sin(\omega t))\\ &= \frac{1}{(9-\omega^2)^2+4\omega^2}\Big((9-\omega^2)\cos(\omega t)+2\omega\sin(\omega t)+i\big((9-\omega^2)\sin(\omega t)-2\omega\cos(\omega t)\big)\Big)\end{aligned}$$

(Going from line 1 to line 2 of this equation would normally be one or two more lines of intermediate computations. Exercise: do it to see where the terms come from.)

So if we take the real part of this, we get:

$$y_p = \frac{1}{(9-\omega^2)^2+4\omega^2}\big((9-\omega^2)\cos(\omega t)+2\omega\sin(\omega t)\big).$$

Not very enlightening, is it?

Here is another method to obtain the real-valued particular solution of the equation from the complex solution. It is recommended, as it gives more meaningful information. First, remark that H is a complex number (which depends on ω). So it has a modulus and an argument. More precisely, we can write:

$$H = |H|e^{i\operatorname{Arg}(H)} = |H|e^{-i\varphi},$$

where $\varphi = -\operatorname{Arg}(H)$ (the reason for the minus sign is that it's the way it is usually presented in physics[1]). We can compute

$$|H| = \frac{1}{\sqrt{(9-\omega^2)+4\omega}}.$$

[1] A sociologist would probably argue that this minus sign is a social construct; a physicist would probably say that "it makes more sense."

This function is noted G and is called the *gain*.

Furthermore, we have seen that $-\text{Arg}(z) = \text{Arg}(1/\text{z})$, therefore

$$\varphi = \text{Arg}(9 - \omega^2 + 2\omega i).$$

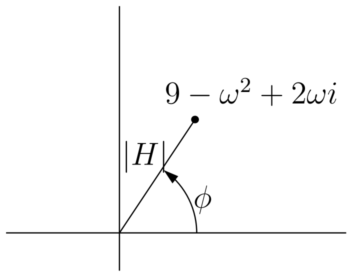


Figure 1: The argument and modulus of H. Review how to find the argument of a complex number using the arctan function, depending on which quadrant the complex number is in.

Now, the particular solution becomes

$$\begin{aligned} z_p &= Ge^{-i\varphi}e^{i\omega t} \\ &= Ge^{i(\omega t - \varphi)} \\ &= \frac{1}{\sqrt{(9-\omega^2)+4\omega}}\big(\cos(\omega t - \varphi) + i\sin(\omega t - \varphi)\big). \end{aligned}$$

And the real solution becomes easy:

$$y_p = \frac{1}{\sqrt{(9-\omega^2)+4\omega}}\cos(\omega t - \varphi)$$

It allows a better interpretation, as we will see below.

General solution of the non-homogeneous equation

In the end, the general solution of the equation is:

1. As a complex function:

$$z = C_1e^{-t}e^{i2\sqrt{2}t} + C_2e^{-t}e^{-i2\sqrt{2}t} + \frac{1}{9-\omega^2+2\omega i}e^{i\omega t}.$$

2. As a real function, version 1:

$$y = C_1e^{-t}\cos(2\sqrt{2}t) + C_2e^{-t}\sin(2\sqrt{2}t) + \frac{1}{(9-\omega^2)^2+4\omega^2}\big((9-\omega^2)\cos(\omega t) + 2\omega\sin(\omega t)\big).$$

3. As a real function, better version:

$$y = C_1e^{-t}\cos(2\sqrt{2}t) + C_2e^{-t}\sin(2\sqrt{2}t) + \frac{1}{\sqrt{(9-\omega^2)+4\omega}}\cos(\omega t - \varphi).$$

Interpretation.

Let us have a look at the solution

$$\begin{aligned} y &= C_1 e^{-t}\cos(2\sqrt{2}t) + C_2 e^{-t}\sin(2\sqrt{2}t) + \frac{1}{\sqrt{(9-\omega^2)+4\omega^2}}\cos(\omega t-\varphi) \\ &= e^{-t}\big(C_1\cos(2\sqrt{2}t) + C_2\sin(2\sqrt{2}t)\big) + G\cos(\omega t-\varphi) \end{aligned}$$

The first term (with e^{-t} as a factor) tends to 0 as t tends to infinity. It is called a *transient* term. It means that after a little while, the solution of the equation will become closer and closer to:

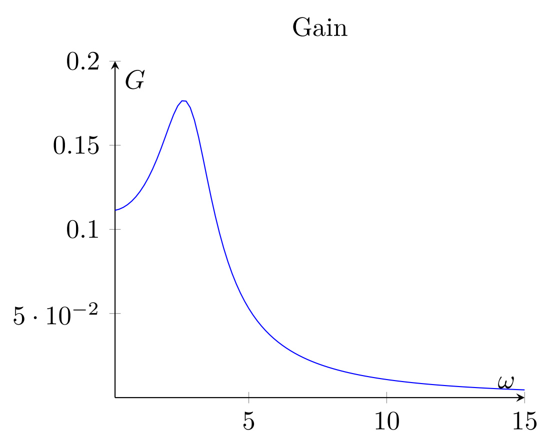
$$y_\infty = G\cos(\omega t-\varphi).$$

This term is called the steady-state term.

The system can be interpreted as follows: the right-hand term of the differential equation $\cos(\omega t)$ corresponds to an *input*: it comes from the outside force that is applied to our system. The solution of the equation is what we observe: it correspond to an *output*. Of course, the output depends on the input. From the form of the steady-state term, we can see that the output[2] is a sinusoid, of the same frequency ω as the input: the system was *forced* to oscillate at this frequency. However, its amplitude is changed (it is multiplied by G), and there is a phase added (a lag), corresponding to φ.

The phase φ can be interpreted as follows: imagine we are dealing with a mass attached to a spring. When the forcing term wants to push the mass in the positive y direction, because of the inertia, it will take a bit of time before the mass actually goes in this direction. This lag between the external forcing and the reaction of the system is measured by φ.

Let's look at G and φ in more details. Below is a plot of G as a function of ω.

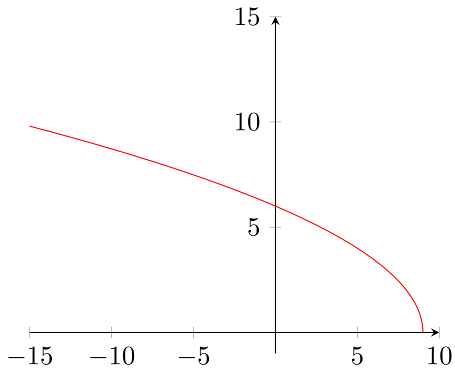


We see that the gain has a maximum. (It is possible to study the function $\omega \mapsto (9-\omega^2)^2+4\omega^2$; the value of ω for which this function is minimal is the one

[2]After the transient term dies out.

for which G is maximum). This specific value of ω corresponds to a frequency of resonance. If the damping were very very low, this frequency of resonance would be approximately $\omega_0 = 3$. Remark also how when ω tends to infinity, the gain tends to 0.

As for φ, let us see at what happens when ω tends to 0 (very slow), and when ω tends to infinity (very high frequency). Let us plot parametrically the set of all $(9 - \omega^2) + 2\omega i$ on the complex plane, in terms of the parameter ω.



We see that it is a parabola. The point on the positive real axis corresponds to $\omega = 0$, and corresponds to $\varphi = 0$. The point on the imaginary axis ($\varphi = \pi/2$) corresponds to $\omega = 3$. As ω tends to infinity, we see that the imaginary part goes much faster to $-\infty$ than the real part goes to $+\infty$. Therefore, φ tends to π as ω tends to $+\infty$.

If the damping is very low (even lower than in this example), the movements of the driving force and the oscillator are in phase when the frequency is low; they are out of phase by $\pi/2$ around the resonance frequency; they are out of phase by π if the frequency is very high. See the lined video.

`http://www.youtube.com/watch?v=aZNnwQ8HJHU`