

# TMA 4115 Matematikk 3

## Lecture 10 for MTFYMA

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In this lecture we discuss...

- introduce vectors
- relate them to linear equations
- discuss operations with vectors (in particular their “span”)

## Solving linear systems

Given a linear system as

$$\begin{aligned}x_1 + 5x_2 + 3x_3 + 2x_4 &= 4 \\x_1 - 2x_3 + 2x_4 &= 0 \\2x_2 + 4x_3 + 2x_4 &= 1\end{aligned}\tag{2}$$

find  $x_1, x_2, x_3, x_4$  which simultaneously satisfy (2).

Use **elementary operations** to replace (2) with an **equivalent** system which is easier.

But first: Rewrite (2) as an **augmented matrix**

$$\begin{bmatrix} 1 & 5 & 3 & 2 & 4 \\ 1 & 0 & -2 & 2 & 0 \\ 0 & 2 & 4 & 2 & 1 \end{bmatrix}$$

Then apply elementary row operations and the Gaussian elimination to obtain a matrix in (reduced) echelon form.

## Row operations & Gaussian elimination

### Elementary row operations:

- Interchange two rows,
- add a multiple of another row to a row, or
- multiply by a non-zero number.

Gaussian elimination uses these operations, to compute (reduced) Echelon form.

**Idea:** Start in the left upper corner and work to the bottom right, using elementary operations.

The steps to compute Echelon form are usually carried out first. We call this the **forward phase** (from left upper corner down). Transforming from Echelon to reduced Echelon form is called the **backward phase** (from right bottom move up, to reduce computation).

## Pivot positions

**Theorem:** For each matrix, there is a unique matrix in reduced echelon form such that both matrices are equivalent via elementary row operations.

**Pivot positions** are positions in a matrix which coincide with a leading one in reduced echelon form

$$\begin{bmatrix} \mathbf{1} & 5 & 3 & 2 & 4 \\ 1 & \mathbf{0} & -2 & 2 & 0 \\ 0 & 2 & \mathbf{4} & 2 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \mathbf{1} & 0 & 0 & 4 & -\frac{6}{10} \\ 0 & \mathbf{1} & 0 & -1 & 1.1 \\ 0 & 0 & \mathbf{1} & 1 & -\frac{3}{10} \end{bmatrix}$$

A **pivot column** is a column with a pivot position.

The reduced echelon matrix yields solutions for (2):

$$x_1 = -\frac{6}{10} - 4x_4, \quad x_2 = x_4 + 1.1, \quad x_3 = -\frac{3}{10} - x_4, \quad x_4 \in \mathbb{R}.$$

## Basic variables and free variables

**7.10 Definition** A variable  $x_i$  of a linear system is called ...

- **basic variable** if it corresponds to a pivot column,
- **free variable** if the associated column does not have a pivot position.

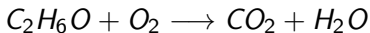
**7.11 Theorem** A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column.

If the linear system is consistent, then there

- (a) is a unique solution if there are no free variables
- (b) infinitely many solutions if there are free variables

# Revisiting a chemical reaction equation

Ethanol + Oxygen  $\longrightarrow$  Carbondioxide + Water



we generated (somehow) the linear equations

$$2x_1 + 0x_2 - 1x_3 - 0x_4 = 0$$

$$6x_1 + 0x_2 - 0x_3 - 2x_4 = 0 \tag{1}$$

$$1x_1 + 2x_2 - 2x_3 - 1x_4 = 0$$

How can we do this in a more systematic way?

**Idea:** The chemical formulae are just lists how many atoms of different kinds are present!

Can write:  $C_2H_6O \rightsquigarrow \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix}$ ,  $O_2 \rightsquigarrow \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$ ,  $CO_2 \rightsquigarrow \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ ,  $H_2O \rightsquigarrow \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

We call the ordered lists obtained in this way **vectors** and naïvely write (1) as follows:

$$x_1 \cdot \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} - x_3 \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - x_4 \cdot \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (2)$$

**Note:** At the moment we have not defined  $+$ ,  $-$ ,  $\cdot$  for vectors!

However, we would like  $+$ ,  $-$ ,  $\cdot$  to be defined such that (2) means the same as equation (1).

# Definition of vectors

## Definition 8.1 (Vector)

A **vector** is an ordered list of numbers  $v_1, v_2, \dots, v_n \in \mathbb{R}$  (or  $\mathbb{C}$ ) for  $n \in \mathbb{N}$ . We write

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Further we let  $\mathbb{R}^n$  be the set of all vectors of length  $n$ .

Special example:  $\vec{0}$  the zero vector (all entries are 0)

To save space we will also often write  $\vec{v} = (v_1, v_2, \dots, v_n)$