TMA 4115 Matematikk 3

Lecture 15 for MTFYMA

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In this lecture we will discuss

- Matrix algebra (i.e. adding and multiplying matrices)
- Inverse matrices

Matrix $A \to$ linear transformation $T_A \colon \mathbb{R}^n \to \mathbb{R}^m, \overrightarrow{x} \mapsto A\overrightarrow{x}$

$T \colon \mathbb{R}^n \to \mathbb{R}^m$ linear $\to$ **standard matrix** $\left[T(\overrightarrow{e_1}) \quad \ldots \quad T(\overrightarrow{e_n}) \right]$

Formulate questions about $A\overrightarrow{x} = \overrightarrow{b}$ in the language of linear transformations. Recall $T \colon \mathbb{R}^n \to \mathbb{R}^m$ is

- **onto** if each $\overrightarrow{b} \in \mathbb{R}^m$ can be written as $\overrightarrow{b} = T(\overrightarrow{x})$ for at least one $\overrightarrow{x}$
- **one-to-one** if each $\overrightarrow{b} \in \mathbb{R}^m$ satisfies $\overrightarrow{b} = T(\overrightarrow{x})$ for at most one $\overrightarrow{x}$

(In the literature: **onto** = **surjective** , **one-to-one** = **injective**)

12. Matrix algebra

Let $f, g\colon \mathbb{R}^n \to \mathbb{R}^m$ be linear and $r \in \mathbb{R}$. Then

$$f + rg\colon \mathbb{R}^n \to \mathbb{R}^m, \overrightarrow{x} \mapsto f(\overrightarrow{x}) + rg(\overrightarrow{x})$$

is linear :

$$\begin{aligned}(f + rg)(\overrightarrow{v} + t\overrightarrow{u}) &= f(\overrightarrow{u} + t\overrightarrow{v}) + rg(\overrightarrow{u} + t\overrightarrow{v}) \\ &= (f + rg)(\overrightarrow{u}) + t(f + rg)(\overrightarrow{v})\end{aligned}$$

For $h\colon \mathbb{R}^m \to \mathbb{R}^p$ linear the composition $h \circ f\colon \mathbb{R}^n \to \mathbb{R}^p$ is linear.

Question:

Are the standard matrices of $(f + rg)$ and $h \circ g$ related to the standard matrices of f, g and h?

Addition and scalar multiplication of matrices

Abbreviate: $A = [a_{ij}]_{\substack{1\le i\le m\\ 1\le j\le n}}$, $m \times n$-matrix with entries a_{ij}

Example (Writing matrices via their entries)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = [a_{ij}]_{\substack{1\le i\le 2\\ 1\le j\le 3}} = [a_{ij}]$$ (Usually we suppress $1 \le i \le m, 1 \le j \le n$)

12.1 Definition:

For $A = [a_{ij}], B = [b_{ij}]$ $m \times n$-matrices and $r \in \mathbb{R}$, define

$$A + B := [a_{ij} + b_{ij}]$$
$$rA := [ra_{ij}]$$

Note: Addition is only defined for matrices of the same size!

Properties of Addition and multiplication

12.2 Lemma

Let f, g be linear transformations and $r \in \mathbb{R}$, then the standard matrix of the linear map $f + rg$ is $A_{f+rg} = A_f + rA_g$.

Proof:
$(f + rg)(\overrightarrow{x}) = f(\overrightarrow{x}) + rg(\overrightarrow{x}) = A_f\overrightarrow{x} + rA_g\overrightarrow{x} = (A_f + rA_g)\overrightarrow{x}$. □

12.3 Rules for addition and scalar multiplication

A, B, C matrices of the same size, $r, s \in \mathbb{R}$, then

- $A + B = B + A$, $(A + B) + C = A + (B + C)$,
- Let 0 be the zero matrix (all entries 0), then $A + 0 = A$,
- $r(A + B) = rA + rB$, $(r + s)A = rA + sA$ and $r(sA) = (rs)A$.

Proof: Easy computations.

Standard matrix for the composition

Let f, g be linear such that $f \circ g$ is defined. For $\overrightarrow{x}$ we compute

$$f \circ g(\overrightarrow{x}) = f(g(x)) = f(A_g \overrightarrow{x}) = A_f(A_g(\overrightarrow{x}))$$

Now if $A_g = \begin{bmatrix} \overrightarrow{a}_1 \dots \overrightarrow{a}_n \end{bmatrix}$ and $\overrightarrow{x} = (x_1, \dots, x_n)$, then we have

$$f \circ g(\overrightarrow{x}) = A_f \left(\sum_{i=1}^{n} x_i \overrightarrow{a}_i \right) = \sum_{i=1}^{n} x_i A_f \overrightarrow{a}_i = \begin{bmatrix} A_f \overrightarrow{a}_1 & \dots & A_f \overrightarrow{a}_p \end{bmatrix} \overrightarrow{x}$$

12.4 Definition

A $m \times n$ matrix, $B = \begin{bmatrix} \overrightarrow{b}_1 & \dots & \overrightarrow{b}_p \end{bmatrix}$ $n \times p$ matrix. Then define

$$AB := A \cdot B := \begin{bmatrix} A\overrightarrow{b}_1 \dots A\overrightarrow{b}_p \end{bmatrix}$$

12.6 Remark

$(m \times n \text{ matrix}) \cdot (n \times p \text{ matrix}) = m \times p$ matrix.

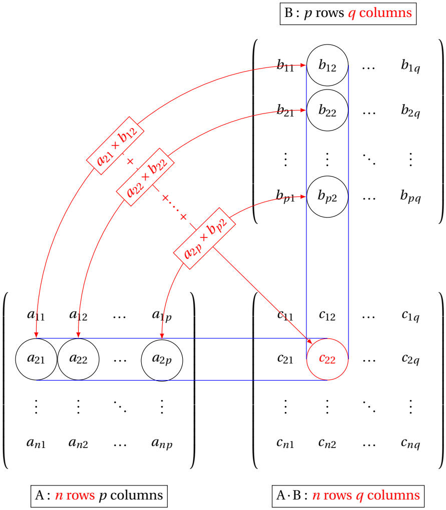
12.7 Rules for matrix multiplication

For matrices A, B, C of suitable size the following rules holds:

- $A(BC) = (AB)C$,
- $A(B+C) = AB + AC$ and $(B+C)A = BA + CA$,
  $r(AB) = (rA)B = A(rB)$
- $I_m A = A = AI_n$ (for A $m \times n$ matrix, I_m, I_n identity matrices)

Proof: $A(BC) = (AB)C$ follows from $(f \circ g) \circ h = f \circ (g \circ h)$.
Rest: Easy computations

Matrix multiplication (diagram from Altermundus.com)



12.8 Row formula for the matrix product

Let $A = [a_{ij}]$ $m \times n$ matrix and $B = [b_{kl}]$ $n \times p$ matrix such that $AB = [c_{rs}]$. Then c_{rs} can be computed by the following formula

$$c_{rs} = a_{r1}b_{1s} + a_{r2}b_{2s} + \ldots + a_{rn}b_{ns}$$

In general $AB \neq BA$! There are $A, B \neq 0$ with $AB = 0$.
Matrix multiplication does not behave like multiplication in $\mathbb{R}$!