TMA 4115 Matematikk 3

Lecture 16 for MTFYMA

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In this lecture we will...

- learn more about invertible matrices,
- introduce the LU-factorisation of matrices,
- study determinants of matrices,

Invertible matrices

A $n \times n$ square matrix A is called **invertible** if there is a square matrix A^{-1} with

$$A \cdot A^{-1} = I_n \quad A^{-1} \cdot A = I_n$$

The equation $A\overrightarrow{x} = \overrightarrow{b}$ for a $n \times n$ matrix A has a unique solution for all $\overrightarrow{b} \in \mathbb{R}^n$ if and only if A is invertible.

Use Gaussian elimination on $\begin{bmatrix} A & I_n \end{bmatrix}$ to compute if A is invertible and if so, get A^{-1}.

12.18 Theorem (Invertible matrix theorem, proof textbook p.130)

The following conditions are equivalent for an $n \times n$-matrix A:

(a) A is invertible,

(b) for all $\overrightarrow{b} \in \mathbb{R}^n$ the equation $A\overrightarrow{x} = \overrightarrow{b}$ has a solution,

(c) the columns of A span $\mathbb{R}^n$,

(d) $T_A \colon \mathbb{R}^n \to \mathbb{R}^n, \overrightarrow{x} \mapsto A\overrightarrow{x}$ is onto,

(e) A has a pivot position in every column,

(f) the reduced echelon form of A is I_n,

(g) $A\overrightarrow{x} = \overrightarrow{0}$ has a unique solution,

(h) the columns of A are linearly independent,

(i) T_A is one-to-one,

(j) there is an $n \times n$ matrix D with $AD = I_n$,

(k) there is an $n \times n$ matrix D with $DA = I_n$

Given a matrix A how can we solve

$$A\overrightarrow{x} = \overrightarrow{b}_1, \quad A\overrightarrow{x} = \overrightarrow{b}_2, A\overrightarrow{x} = \overrightarrow{b}_3 \ldots$$

efficiently?

If A is invertible, we could compute A^{-1} and then $A^{-1}\overrightarrow{b}_1, \ldots$

However, it is faster to try something different.

Example: LU factorisation of a matrix A

$$\begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

We want to solve $A\overrightarrow{x} = \begin{bmatrix} -9 & 5 & 7 & 11 \end{bmatrix}^T$. Hence use LU factorisation and compute

$$L\overrightarrow{y} = \overrightarrow{b}$$
$$U\overrightarrow{x} = \overrightarrow{y}$$

$$\begin{bmatrix} L & \overrightarrow{b} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ -1 & 1 & 0 & 0 & 5 \\ 2 & -5 & 1 & 0 & 7 \\ -3 & 8 & 3 & 1 & 11 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} I & \overrightarrow{y} \end{bmatrix}$$

Thus $\overrightarrow{y} = \begin{bmatrix} -9 & -4 & 5 & 1 \end{bmatrix}^T$. Now solve

$$\begin{bmatrix} U & \overrightarrow{y} \end{bmatrix} = \begin{bmatrix} 3 & -7 & -2 & 2 & -9 \\ 0 & -2 & -1 & 2 & -4 \\ 0 & 0 & -1 & 1 & 5 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} I & \overrightarrow{x} \end{bmatrix}$$

Hence $\overrightarrow{x} = \begin{bmatrix} 3 & 4 & -6 & -1 \end{bmatrix}^T$.

How to compute an LU factorisation

LU factorisation for matrix A

- Reduce A to Echelon form U using row replacement operations only (if possible)
- Place entries in L such that the same row operations reduce L to I.

The first step in the above algorithm is not always possible even if A is invertible.

One can prove that A invertible can always be decomposed as $A = PLU$, where the matrix P just interchanges rows.

Invertible 2×2 matrices

Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Try to compute an inverse:

$$\begin{bmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} a & b & 1 & 0 \\ 0 & \frac{ad-bc}{a} & -\frac{c}{a} & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & \frac{-c}{ad-bc} & \frac{b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

Observation:

A is invertible if the **determinant** $\det A = ad - bc$ is not zero.

Similar for 2nd order differential equations: Wronskian is the determinant of a matrix (with functions as entries).

Question:

Can we define a “determinant” for arbitrary square matrices?

The 3×3 case

Row reduce an (invertible) 3×3 matrix (with $a_{11} \neq 0$):

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightsquigarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & 0 & a_{11}\Delta \end{bmatrix}$$

$$\text{with } \Delta = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.$$

Now A can only be invertible, if $\Delta \neq 0$.

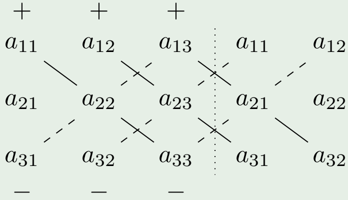
3×3 determinants

We define

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \Delta = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$- a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.$$

To remember this formula, use the "rule of Sarrus":



How to produce a general formula for the $n \times n$-case?

Rewrite Δ for the 3×3 matrix as

$$\begin{aligned}
\Delta =& (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}) + (a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33}) \\
&+ (a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}) \\
=& a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) \\
&+ a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\
=& a_{11}\det \begin{bmatrix} a_{22} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} - a_{12}\det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det \begin{bmatrix} a_{21} & a_{22} \\ a_{32} & a_{31} \end{bmatrix}
\end{aligned}$$