TMA 4115 Matematikk 3

Lecture 17 for MTFYMA

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In today's lecture we discuss

- Determinants (Definition and how to compute them)
- "Abstract" vector spaces

Determinants

The *determinant* of a matrix A tells us if a matrix is invertible!
(Recall $\det A \neq 0$ if A is invertible)

Example determinant of a 2×2 matrix

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

Determinant of a 3×3 matrix

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Recursive Definition of Determinants

14.1 Definition

A a $n \times n$ matrix, $1 \leq i, j \leq n$.
Define a matrix A_{ij} by deleting in A the ith row and jth column.

14.2 Definition (**Determinant** of the matrix $A = \left[a_{ij}\right]_{1\leq i,j\leq n}$)

For $n = 1$: $\det \left[a_{11}\right] = a_{11}$.
For $n \geq 2$ define the determinant as

$$\begin{aligned} \det A &= a_{11}\det A_{11} - a_{12}\det A_{12} + \cdots + a_{1n}\det A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j}\det A_{1j} \end{aligned}$$

An interesting observation

Linear differential equations

$$y'' + 2y' + y = g(x)$$

Linear systems

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ 42x_1 - x_3 &= 12 \\ x_2 + x_3 &= 1 \end{aligned}$$

Goal

Find **functions** solving the equation

Goal

Find **numbers/vectors** solving the system

Both topics are connected! (Determinants (/Wronskian), general solutions, homogeneous equations...)

Question

What is the theoretical explanation?

Reformulating Linear systems

Linear system

$$x_1 + 2x_2 - x_3 = 0$$
$$42x_1 - x_3 = 12$$
$$x_2 + x_3 = 1$$

$\leftrightarrow$

Matrix equation

$$A\overrightarrow{x} = \begin{bmatrix} 0 \\ 12 \\ 1 \end{bmatrix}$$

Matrix A

$$\begin{bmatrix} 1 & 2 & -1 \\ 42 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$\leftrightarrow$

Linear Equation

$$T_A(\overrightarrow{x}) = \begin{bmatrix} 0 \\ 12 \\ 1 \end{bmatrix}$$

linear/matrix transformation

$T_A \colon \mathbb{R}^3 \to \mathbb{R}^3$.

Question

Can we view a linear differential equation in the same way?

Linear transformations (on functions?)

Consider the 2nd order equation $\underbrace{y'' + 2y' + y}_{=: T(y)} = g(t)$

Then T is a transformation for functions:

y	$T(y) = y'' + 2y' + y$
e^t	$4e^t$
$\sin(t)$	$2\cos(t)$
$\cos(t)$	$-2\sin(t)$
t	t
t^2	$2 + 2t + t^2$

T is even "linear" (Functions behave in this example like vectors!):

y	$T(y) = y'' + 2y' + y$
$e^t + \sin(t)$	$4e^t + 2\cos(t)$
$t + t^2 + \cos(t)$	$t + 2 + 2t + t^2 - 2\sin(t)$
0	0

Functions as vectors?

Functions $f, g, h\colon \mathbb{R} \to \mathbb{R}$ can be added and multiplied pointwise:

- $(f+g)(t) := f(t) + g(t) = (g+f)(t)$
- $(f+g)+h = f+(g+h)$
- Let 0 be the function which is constant 0, then $0 + f = f = f + 0$
- $r \cdot (s \cdot f) = (rs) \cdot f = s \cdot (r \cdot f)$
- $(r+s)f = r \cdot f + s \cdot f$
- $r \cdot (f+g) = r \cdot f + r \cdot g$
- $f + (-1) \cdot f = f - f = 0$
- $1 \cdot f = f$

15.1 Definition (abstract) vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$

A ($\mathbb{K}$-)**vector space** is a non-empty set V of objects (= **vectors**), with "+" *addition* and
"$\cdot$" *multiplication* by **scalars** (=numbers $\mathbb{K}$).
such that for vectors $\overrightarrow{u}$, $\overrightarrow{v}$, $\overrightarrow{w}$ in V and $r, s \in \mathbb{K}$ the following holds:

- $\overrightarrow{u} + \overrightarrow{v} = \overrightarrow{v} + \overrightarrow{u}$ and $(\overrightarrow{u} + \overrightarrow{v}) + \overrightarrow{w} = \overrightarrow{u} + (\overrightarrow{v} + \overrightarrow{w})$
- There is a **zero-vector** $\overrightarrow{0}$ with $\overrightarrow{0} + \overrightarrow{v} = \overrightarrow{v}$
- for $\overrightarrow{v}$ there is $-\overrightarrow{v}$ with $\overrightarrow{v} + (-\overrightarrow{v}) = \overrightarrow{0}$ $((-1) \cdot \overrightarrow{v} = -\overrightarrow{v})$
- $r(\overrightarrow{u} + \overrightarrow{v}) = r\overrightarrow{u} + r\overrightarrow{v}$
- $(r + s)\overrightarrow{v} = r\overrightarrow{v} + s\overrightarrow{v}$
- $(rs)\overrightarrow{v} = r(s\overrightarrow{v})$
- $1\overrightarrow{v} = \overrightarrow{v}$

Normally consider only $\mathbb{R}$-vector spaces, so $\mathbb{K} = \mathbb{R}$.

Vector spaces

(Most important) Example

$\mathbb{R}^n$ is a vector space.

If you have trouble with abstract vector spaces, always think of how things work in $\mathbb{R}^n$!

Vector spaces generalize $\mathbb{R}^n$. Thus they show

- what properties in $\mathbb{R}^n$ are important for linear algebra,
- how to extend linear algebra to more general situations.