TMA 4115 Matematikk 3

Lecture 18 for MTFYMA

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In today's lecture we discuss

- linear algebra in (abstract) vector spaces
- familiar concepts (span, linear transformations) in abstract vector spaces
- row, column and nullspace of a matrix

(Abstract) vector space

Fix $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A ($\mathbb{K}$-)**vector space** is a non-empty set V of objects, called **vectors**, with operations "$+$" *addition* and "$\cdot$" *multiplication* by **scalars** (=numbers in $\mathbb{K}$).

Vector subspace

A **subspace** H of V is a subset $H \subseteq V$ such that

- $\overrightarrow{0} \in H$,
- for $\overrightarrow{v}, \overrightarrow{w} \in H$ and $r \in \mathbb{K}$ the sum $\overrightarrow{v} + r\overrightarrow{w} \in H$.

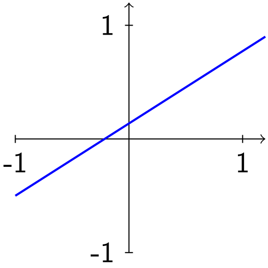
Idea: Vector spaces behave like $\mathbb{R}^n$ and the many important examples arise as subspaces of $\mathbb{R}^n$.

15.4 Examples: Subspaces of $\mathbb{R}^2$

- $\mathbb{R}^2, \{ \overrightarrow{0} \}$ are subspaces of $\mathbb{R}^2$.
- $\overrightarrow{x} \in \mathbb{R}^2$ the span $\{ \overrightarrow{x} \}$ is a subspace of $\mathbb{R}^2$.

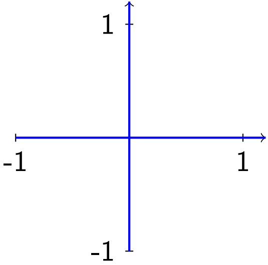
Subsets of $\mathbb{R}^2$ which are not subspaces:

$\{ (1, -0.5) + r(2.2, 1.4) | r \in \mathbb{R} \}$



Does not contain $\overrightarrow{0}$!

$U = \{ (x, y) \in \mathbb{R}^2 | xy = 0 \}$



$(1, 0), (0, 1) \in U$ but
$(1, 0) + (0, 1) = (1, 1) \notin U$!

More (Counter-)Examples of subspaces

- For $k > l$ we have $C^k(\mathbb{R}, \mathbb{R})$ is a subspace of $C^l(\mathbb{R}, \mathbb{R})$.[1]
- $\mathbb{P}_n$ (Polynomials up to order n) is a subspace of every $C^k(\mathbb{R}, \mathbb{R})$.
- $\mathbb{R}^2$ is **not** a subspace of $\mathbb{R}^3$!

[1] $C^k(\mathbb{R}, \mathbb{R})$vector space of $k$times continuously differentiable maps $\mathbb{R} \to \mathbb{R}$

Vector spaces are designed to have the features of $\mathbb{R}^n$ which enable linear algebra!

Motto

Copy old definitions from $\mathbb{R}^n$ to arbitrary vector spaces!

Familiar concepts now in vector spaces I

Definition

Let V be a vector space and $\overrightarrow{v_1}, \ldots, \overrightarrow{v_k} \in V$.

- A **linear combination** of $\overrightarrow{v_1}, \ldots, \overrightarrow{v_k}$ is a weighted sum

$$\sum_{l=1}^{k} r_l \overrightarrow{v_l} = r_1 \overrightarrow{v_1} + \ldots + r_k \overrightarrow{v_k}$$

  span $\{ \overrightarrow{v_1}, \ldots, \overrightarrow{v_k} \}$ = **set of all linear combinations** of $\overrightarrow{v_1}, \ldots, \overrightarrow{v_k}$.

- $\overrightarrow{v_1}, \ldots, \overrightarrow{v_k}$ are called **linearly independent** if

$$\sum_{i=1}^{k} r_i \overrightarrow{v}_i = r_1 \overrightarrow{v}_1 + r_2 \overrightarrow{v}_2 + \ldots + r_k \overrightarrow{v}_k = \overrightarrow{0}$$

  has only the trivial solution $r_1 = r_2 = \cdots = r_k = 0$.

Familiar concepts now in vector spaces II

Definition

Let V, W be vector spaces. A function $T\colon V \to W$ is called a **linear transformation** if for all vectors $\overrightarrow{v}, \overrightarrow{w}$ and each scalar $r \in \mathbb{K}$ the following holds

$$T(\overrightarrow{u} + r\overrightarrow{v}) = T(\overrightarrow{u}) + rT(\overrightarrow{v})$$

Examples

- A a $m \times n$ matrix, the matrix transformation $T_A\colon \mathbb{R}^n \to \mathbb{R}^n, \overrightarrow{x} \mapsto A\overrightarrow{x}$ is linear
- $T\colon C^\infty(\mathbb{R},\mathbb{R}) \to C^\infty(\mathbb{R},\mathbb{R}), f \mapsto f'$ is linear.
- $\mathrm{ev}_0\colon C^\infty(\mathbb{R},\mathbb{R}) \to \mathbb{R}, f \mapsto f(0)$ is linear.
- $q\colon C^1(\mathbb{R},\mathbb{R}) \to C^1(\mathbb{R},\mathbb{R}), f \mapsto \cos \circ f$ is not linear.