TMA 4115 Matematikk 3

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In today's lecture we discuss...

- linear equations and their relations to vector and matrix equations
- solution sets of linear equations

Vectors

Vectors $\approx$ ordered list of numbers.
$\mathbb{R}^n$ set of vectors of length n
(i.e. vectors with n entries from $\mathbb{R}$).

Have operations $+, -$ for vectors and $r \cdot \overrightarrow{v}$ for $r \in \mathbb{R}$ (or $\mathbb{C}$).

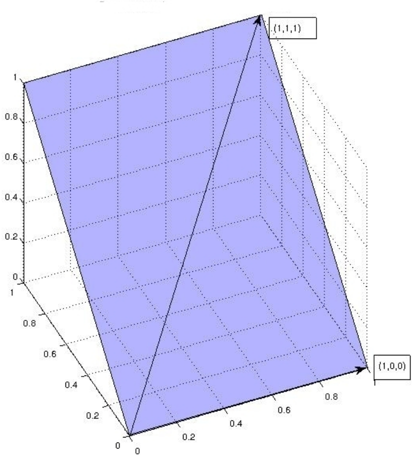
linear combination of vectors

$\overrightarrow{v_1}, \ldots, \overrightarrow{v_k} \in \mathbb{R}^n$ is a weighted sum

$$\sum_{l=1}^{k} r_l \overrightarrow{v_l} = r_1 \overrightarrow{v_1} + \ldots + r_k \overrightarrow{v_k}$$

span $\{ \overrightarrow{v_1}, \ldots, \overrightarrow{v_k} \}$ = **set of all linear combinations** of $\overrightarrow{v_1}, \ldots, \overrightarrow{v_k}$

A picture of span $\{(1,1,1),(1,0,0)\}$



The vectors are not multiples of each other (and both are not $\vec{0}$), so they span a plane in $\mathbb{R}^3$.

Linear Equations in chemistry

We want to balance the reaction equation

$$\text{Ethanol} + \text{Oxygen} \longrightarrow \text{Carbondioxide} + \text{Water}$$
$$C_2H_6O + O_2 \longrightarrow CO_2 + H_2O$$

Introduce indeterminates x_1, x_2, x_3, x_4 and write

$$x_1 C_2H_6O + x_2 O_2 = x_3 CO_2 + x_4 H_2O$$

Note: $x_1 C_2H_6O$ has $2x_1$ atoms of carbon, $6x_1$ atoms of hydrogen and x_1 atoms of oxygen.

Idea

To generate a system of linear equations we generate an equation for each type of atom in the equation.

Linear Equations in chemistry

$$x_1 C_2H_6O + x_2 O_2 = x_3 CO_2 + x_4 H_2O$$

Molecules give rise to vectors!

We write: $C_2H_6O \rightsquigarrow \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix}$, $O_2 \rightsquigarrow \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$, $CO_2 \rightsquigarrow \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $H_2O \rightsquigarrow \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

Thus the equation becomes

$$x_1 \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Linear Equations in chemistry

We solve the system with Gaussian elimination

$$\begin{bmatrix} 2 & 0 & -1 & 0 & 0 \\ 6 & 0 & 0 & -2 & 0 \\ 1 & 2 & -2 & -1 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & -2 & 0 \\ 0 & 2 & -\frac{3}{2} & -1 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & 0 \end{bmatrix}$$

Parametric vector form of the solution is thus

$$x_4(\frac{1}{3}, 1, \frac{2}{3}, 1)$$

Note that we should have $x_1, x_2, x_3, x_4 \in \mathbb{N}$. Choose the smallest x_4 such that this is the case, i.e. $x_4 = 3$ then

$$1C_2H_6O + 3O_2 = 2CO_2 + 3H_2O$$

is balanced.

Linear systems vs. vector equations vs. matrices

$$\begin{aligned} x_1 + 5x_2 + 3x_3 + 2x_4 &= 4 \\ x_1 - 2x_3 + 2x_4 &= 0 \\ 2x_2 + 4x_3 + 2x_4 &= 1 \end{aligned}$$

can be rewritten as a **vector equation**

$$x_1 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix} + x_3 \cdot \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} + x_4 \cdot \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$$

Solutions to the linear system $\leftrightarrow$ solutions to the vector equation.

Also the linear system is **represented** by the augmented matrix

$$\begin{bmatrix} 1 & 5 & 3 & 2 & 4 \\ 1 & 0 & -2 & 2 & 0 \\ 0 & 2 & 4 & 2 & 1 \end{bmatrix}$$

Linear systems vs. vector equations vs. matrices II

The augmented matrix is used to solve the linear system but the matrix representation is a *representation* and not an *equation*.

Question: Can we rewrite the linear system as a matrix equation?
Idea: Compare the coefficients matrix of the linear system

$$A = \begin{bmatrix} 1 & 5 & 3 & 2 \\ 1 & 0 & -2 & 2 \\ 0 & 2 & 4 & 2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix} & \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} & \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \end{bmatrix}$$

with the vector equation

$$x_1 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix} + x_3 \cdot \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} + x_4 \cdot \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$$

Linear systems vs. vector equations vs. matrices III

Then we define for $\overrightarrow{x} = (x_1, x_2, x_3, x_4)$ the equation $A\overrightarrow{x} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$ to mean the same as the vector equation

$$x_1 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix} + x_3 \cdot \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} + x_4 \cdot \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$$

In fact we can use this principle to define a product $A\overrightarrow{x}$ for any suitable vector $\overrightarrow{x}$!