TMA 4115 Matematikk 3

Lecture 19 for MTFYMA

Alexander Schmeding

NTNU

14. March 2016

In today's lecture we will

- learn more about vector space bases
- ... and how they induce coordinate systems
- see an example of how this makes abstract vector spaces useful

Basis of a vector space

Let V be a vector space. A subset $\mathcal{B} = \{ \overrightarrow{b}_1, \ldots, \overrightarrow{b}_n \}$ is called **basis** of V if

- span $\{ \overrightarrow{b}_1, \ldots, \overrightarrow{b}_n \} = V$
- the set $\mathcal{B}$ is linearly independent

We can think of a basis as a “minimal” system generating V.

Example

The **standard basis** for $\mathbb{R}^n$, are the unit vectors $\overrightarrow{e}_1, \ldots \overrightarrow{e}_n \in \mathbb{R}^n$

Polynomial basis $\mathcal{P} = \{1, t, t^2, \ldots, t^n\}$ for $\mathbb{P}_n$.

$H = \text{span}\{\overrightarrow{v}_1, \overrightarrow{v}_2, \ldots, \overrightarrow{v}_n\}$ subspace of a vector space V.

Strategies to construct a basis for H

- Remove (step-by-step) vectors from $\{\overrightarrow{v}_1, \overrightarrow{v}_2, \ldots, \overrightarrow{v}_n\}$ which are linear combinations of the other vectors.
  $\rightsquigarrow$ Build a linear independent generating set.
  (For $V = \mathbb{R}^n$ use Gaussian elimination!)
- Start with $\overrightarrow{0} \neq \overrightarrow{v} \in \{\overrightarrow{v}_1, \overrightarrow{v}_2, \ldots, \overrightarrow{v}_n\}$ and add vectors from $\{\overrightarrow{v}_1, \overrightarrow{v}_2, \ldots, \overrightarrow{v}_n\}$, make sure in every step that the span gets larger!

Definition

Let V, W be vector spaces. A function $T\colon V \to W$ is called a **linear transformation** if for all $\overrightarrow{v}, \overrightarrow{w} \in V$ and $r \in \mathbb{K}$ we have

$$T(\overrightarrow{u} + r\overrightarrow{v}) = T(\overrightarrow{u}) + rT(\overrightarrow{v})$$

Define

kernel of T: $\ker T = \{\overrightarrow{v} \in V | T(\overrightarrow{v}) = 0\}$

image of T: $\operatorname{im} T = \{\overrightarrow{w} \in W | \exists \overrightarrow{x} \in V \text{ with } T(\overrightarrow{x}) = \overrightarrow{w}\}$

Examples

- A a $m \times n$ matrix, $T_A\colon \mathbb{R}^n \to \mathbb{R}^n, \overrightarrow{x} \mapsto A\overrightarrow{x}$ is linear, $\ker T_A = \text{Nul}(A)$ and $\operatorname{im} T_A = \text{Col}\,(A)$.
- $\text{ev}_0\colon C^\infty(\mathbb{R}, \mathbb{R}) \to \mathbb{R}, f \mapsto f(0)$ is linear, $\ker \text{ev}_0 = \{f \in C^\infty(\mathbb{R}, \mathbb{R}), f(0) = 0\}$ and $\operatorname{im} \text{ev}_0 = \mathbb{R}$.

Coordinate systems from bases

Let $\mathcal{B} = \{\overrightarrow{b}_1, \ldots, \overrightarrow{b}_n\}$ be a basis of a vector space V then each $\overrightarrow{x} \in V$ can be written as a unique linear combination

$$\overrightarrow{x} = \sum_{i=1}^{n} c_i \overrightarrow{b}_i$$

Obtain an invertible (!) linear transformation

$$K_{\mathcal{B}} \colon V \to \mathbb{R}^n, \overrightarrow{x} = \sum_{i=1}^{n} c_i \overrightarrow{b}_i \mapsto (c_1, c_2, \ldots, c_n)$$

Coordinates for $\mathbb{P}_n$ (Polynomials up to degree n)

Recall the Polynomial basis $\mathcal{P} = \{1, t, t^2, \ldots, t^n\}$ for $\mathbb{P}_n$.
Then

$$K_{\mathcal{P}} \colon \mathbb{P}_n \to \mathbb{R}^{n+1}, \sum_{i=0}^{n} a_i t^i \mapsto [a_0 \ a_1 \ \ldots \ a_n]^T.$$

Translating problems to $\mathbb{R}^n$

Finding an unknown polynomial

In an experiment we observe the following values of an unknown function f:

time t	0	1	2	3
$f(t)$	.4	1.2	$-.2$	0

Can we approximate f with something simple, i.e. is there a polynomial of (at most) degree 3 which takes these values?

Idea: Use linear functions and "translate the problem" from $\mathbb{P}_3$ to a problem written in linear functions between $\mathbb{R}^n$ and $\mathbb{R}^m$!

Translating problems to $\mathbb{R}^n$

Our aim is now to rewrite

time t	0	1	2	3
$f(t)$	.4	1.2	$-.2$	0

Idea: Use the linear functions

$$\mathrm{ev}_k\colon \mathbb{P}_3 \to \mathbb{R}, \quad p(t) \mapsto p(k), \quad k = 0, 1, 2, 3.$$

then:

$$\mathrm{ev}_0(f) = .4, \mathrm{ev}_1(f) = 1.2, \mathrm{ev}_2(f) = -.2 \text{ and } \mathrm{ev}_3(f) = 0. \qquad (1)$$

Let us get rid of $\mathbb{P}_3$ in (1). **Idea**: The coordinate function $K_{\mathcal{P}}\colon \mathbb{P}_3 \to \mathbb{R}^4$ of the polynomial basis is invertible, with inverse

$$S\colon \mathbb{R}^4 \to \mathbb{P}_3, \quad \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \end{bmatrix} \mapsto \sum_{i=0}^{3} a_i t^i$$

Translating problems to $\mathbb{R}^n$

To find the polynomial we search, we can thus:

Find $\overrightarrow{x} \in \mathbb{R}^4$ such that $\overrightarrow{x}$ satisfies the system of equations

$$\mathrm{ev}_0 \circ S(\overrightarrow{x}) = .4 \qquad \mathrm{ev}_1 \circ S(\overrightarrow{x}) = 1.2$$
$$\mathrm{ev}_2 \circ S(\overrightarrow{x}) = -.2 \qquad \mathrm{ev}_3 \circ S(\overrightarrow{x}) = 0$$

Then use S to "translate" $\overrightarrow{x} \in \mathbb{R}^4$ to a polynomial.

As ev_k and S are linear transformations, the functions

$$\mathrm{ev}_k \circ S \colon \mathbb{R}^4 \to \mathbb{R}, \quad \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \end{bmatrix}^T \mapsto \sum_{i=0}^{3} a_i k^i$$

are linear, whence matrix transformations! (Can compute standard matrices)

Translating problems to $\mathbb{R}^n$

Computing standard matrices for the functions $\mathrm{ev}_k \circ S$ we have:

$$A_{\mathrm{ev}_0 \circ S} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \qquad A_{\mathrm{ev}_1 \circ S} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

$$A_{\mathrm{ev}_2 \circ S} = \begin{bmatrix} 1 & 2 & 4 & 8 \end{bmatrix} \qquad A_{\mathrm{ev}_3 \circ S} = \begin{bmatrix} 1 & 3 & 9 & 27 \end{bmatrix}$$

Thus we can find the polynomial as follows:

Solve the matrix equation

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{bmatrix} \cdot \overrightarrow{x} = \begin{bmatrix} .4 \\ 1.2 \\ -.2 \\ 0 \end{bmatrix} \qquad (2)$$

Then $S(\overrightarrow{x}) = K_{\mathcal{P}}^{-1}(\overrightarrow{x})$ is the polynomial we seek!

Solving the matrix equation (2) we obtain $\overrightarrow{x} = \begin{bmatrix} .4 \\ \frac{19}{6} \\ -3 \\ \frac{19}{30} \end{bmatrix}$

Hence $f(t) = S(\overrightarrow{x}) = .4 + \frac{19}{6}t - 3t^2 + \frac{19}{30}t^3$ is a polynomial which satisfies

time t	0	1	2	3
$f(t)$	.4	1.2	−.2	0