TMA 4115 Matematikk 3

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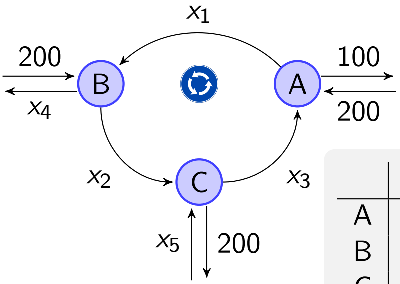
In this lecture we discuss

- Linear systems arising from linear networks
- Linear (in-)dependence of vectors...
- ... what this means for solutions of linear systems

Example: Traffic flow in a roundabout

We count cars in a roundabout[1].



A	$200 + x_3 = 100 + x_1$
B	$x_1 + 200 = x_2 + x_4$
C	$x_5 + x_2 = 200 + x_3$

Question: How can we model the traffic in the roundabout?

[1] Note that in a roundabout cars are only allowed to travel in one direction.

The linear system gives rise to the augmented matrix

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 100 \\ -1 & 1 & 0 & 1 & 0 & 200 \\ 0 & 1 & -1 & 0 & 1 & 200 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 100 \\ 0 & 1 & -1 & 0 & 1 & 200 \\ 0 & 0 & 0 & 1 & -1 & 100 \end{bmatrix}$$

Thus the parametric vector form of the general solution is

$$\begin{bmatrix} 100 \\ 200 \\ 0 \\ 100 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Solutions of matrix equations

We were interested in the following questions about $A\overrightarrow{x} = \overrightarrow{b}$

1. Is there a solution at all (i.e. is the system consistent)?
2. Is there a unique solution?

For $A = [\overrightarrow{a_1}\ \overrightarrow{a_2}\ \dots\ \overrightarrow{a_n}]$ question 1 can be rephrased as:

- Is $\overrightarrow{b} \in \text{span}\{\overrightarrow{a_1}, \dots, \overrightarrow{a_n}\}$?

What about the second question?

Idea: The general solution of $A\overrightarrow{x} = \overrightarrow{b}$ will produce a unique solution if $A\overrightarrow{x} = \overrightarrow{0}$ has a unique solution.

The trivial solution $\overrightarrow{0}$ always exists. Hence, $A\overrightarrow{x} = \overrightarrow{0}$ has a unique solution if and only if $\overrightarrow{0}$ is the only solution.

Linear Independence

A family of vectors $\overrightarrow{v}_1, \ldots, \overrightarrow{v}_k$ are called

- linearly independent if

$$\sum_{i=1}^{n} r_i \overrightarrow{v}_i = \overrightarrow{0} \tag{1}$$

  only admits the trivial solution $r_1 = r_2 = \ldots = r_n = 0$.
- If (1) admits a non trivial solution (at least one $r_i \neq 0$) then the vectors are linearly dependent (call (1) linear dependency relation).

Test for linear independence

Use Gaussian elimination on the matrix $[\overrightarrow{v}_1 \ \overrightarrow{v}_2 \ldots \overrightarrow{v}_k]$. The vectors are linearly independent if and only if every column is a pivot column.

Summary: span and linear (in-)dependence

Let $\overrightarrow{v_1}, \ldots, \overrightarrow{v_p}$ be vectors and $A = \begin{bmatrix} \overrightarrow{v_1} & \overrightarrow{v_2} & \ldots & \overrightarrow{v_p} \end{bmatrix}$.

- span$\{\overrightarrow{v_1}, \ldots, \overrightarrow{v_p}\}$ is the set of all vectors which can be generated from $\overrightarrow{v_1}, \ldots, \overrightarrow{v_p}$.

Test for $\overrightarrow{b} \in \text{span}\{\overrightarrow{v_1}, \ldots, \overrightarrow{v_p}\}$

If $A\overrightarrow{x} = \overrightarrow{b}$ is consistent, $\overrightarrow{b}$ is in the span.

- Linear (in-)dependence is about redundancy, i.e. whether we can combine vectors from fewer vectors.

Test for linear (in-)dependence of $\{\overrightarrow{v_1}, \ldots, \overrightarrow{v_p}\}$

If $[A\,\overrightarrow{0}\,]$ contains only basic variables, the family is linearly independent. Otherwise, the family is linearly dependent.

Moral: Use Gaussian elimination to settle these questions!