TMA 4115 Matematikk 3

Lecture 20 for MTFYMA

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Coordinate systems from bases

Let $\mathcal{B} = \{\overrightarrow{b}_1, \ldots, \overrightarrow{b}_n\}$ be a basis of a vector space V then each $\overrightarrow{x} \in V$ can be written as a unique linear combination

$$\overrightarrow{x} = \sum_{i=1}^{n} c_i \overrightarrow{b}_i$$

Obtain an invertible (!) linear transformation

$$K_{\mathcal{B}} \colon V \to \mathbb{R}^n, \overrightarrow{x} = \sum_{i=1}^{n} c_i \overrightarrow{b}_i \mapsto (c_1, c_2, \ldots, c_n)$$

Coordinates for $\mathbb{P}_3$ (Polynomials up to degree 3)

Polynomial basis $\mathcal{P} = \{1, t, t^2, t^3\}$.
Then

$$K_{\mathcal{P}} \colon \mathbb{P}_3 \to \mathbb{R}^4, \sum_{i=0}^{3} a_i t^i \mapsto [a_0 \;\; a_1 \;\; a_2 \;\; a_3]^T.$$

Translating problems to $\mathbb{R}^n$

Finding an unknown polynomial

In an experiment we observe the following values of an unknown function f:

time t	0	1	2	3
$f(t)$	.4	1.2	$-.2$	0

Can we approximate f with something simple, i.e. is there a polynomial of (at most) degree 3 which takes these values?

Idea: Rewrite the table using the linear functions

$$\mathrm{ev}_k : \mathbb{P}_3 \to \mathbb{R}, \quad p(t) \mapsto p(k), \quad k = 0, 1, 2, 3.$$

$$\mathrm{ev}_0(f) = .4, \mathrm{ev}_1(f) = 1.2, \mathrm{ev}_2(f) = -.2 \text{ and } \mathrm{ev}_3(f) = 0. \tag{1}$$

To get rid of $\mathbb{P}_3$ in (1) use

$$S = K_{\mathcal{P}}^{-1} \colon \mathbb{R}^4 \to \mathbb{P}_3, \quad \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \end{bmatrix}^T \mapsto \sum_{i=0}^{3} a_i t^i$$

and note that the functions

$$\mathrm{ev}_k \circ S \colon \mathbb{R}^4 \to \mathbb{R}, \quad \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \end{bmatrix}^T \mapsto \sum_{i=0}^{3} a_i k^i$$

are linear, whence matrix transformations! (Can compute standard matrices)

Translating problems to $\mathbb{R}^n$

Computing standard matrices for the functions $\mathrm{ev}_k \circ S$ we have:

$$A_{\mathrm{ev}_0 \circ S} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \qquad A_{\mathrm{ev}_1 \circ S} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

$$A_{\mathrm{ev}_2 \circ S} = \begin{bmatrix} 1 & 2 & 4 & 8 \end{bmatrix} \qquad A_{\mathrm{ev}_3 \circ S} = \begin{bmatrix} 1 & 3 & 9 & 27 \end{bmatrix}$$

Thus we can find the polynomial as follows:

Solve the matrix equation

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{bmatrix} \cdot \overrightarrow{x} = \begin{bmatrix} .4 \\ 1.2 \\ -.2 \\ 0 \end{bmatrix} \tag{2}$$

Then $S(\overrightarrow{x}) = K_{\mathcal{P}}^{-1}(\overrightarrow{x})$ is the polynomial we seek!

Solving the matrix equation (2) we obtain $\overrightarrow{x} = \begin{bmatrix} .4 \\ \frac{19}{6} \\ -3 \\ \frac{19}{30} \end{bmatrix}$

Hence $f(t) = S(\overrightarrow{x}) = .4 + \frac{19}{6}t - 3t^2 + \frac{19}{30}t^3$ is a polynomial which satisfies

time t	0	1	2	3
$f(t)$	.4	1.2	$-.2$	0

Two problems from the exams

Autumn 2008 5b:

Assume that each year 30% of owners of cars with two-wheel drive change to a car with four-wheel drive, whilst 10% of owners of cars with four-wheel drive change to a car with two-wheel drive. The total number of cars is constant, and each car owner has only one car. Given that 25% of car owners have four-wheel drive now, what percentage of car ownwers will have four-wheel drive in ten years' time?

Two problems from the exams II

Spring 2011 6b:

There are two places in Trondheim with bicycles that can be hired for free: Gløshaugen (G) and Torget (T). The bicycles can be hired from early in the morning and must be returned to one of the places the same evening. It is found that of the bicycles hired from G, 80% are returned to G and 20% to T. Of the bicycles hired from T, 30% are returned to G and 70% to T. We assume that this pattern is constant, that all bicycles are hired out each morning, and that no bicycles are stolen. In the long term, what proportion of the bicycles will be at Gløshaugen each morning?

Similarities

- The population is divided into a finite set of mutually exclusive states.
- The system evolves in discrete time intervals and in each interval the individuals can change state.
- An individual changes state according to a set list of probabilities that depends only on the current state and is independent of time.

This situation often occurs when we model (dynamical) systems in the natural sciences!

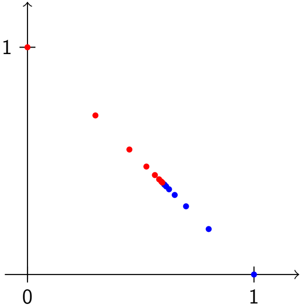
We call such systems **Markov chains**.

15.3 Spring 2011 6 b
Example for a Markov chain

$$P = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix},$$

$\overrightarrow{x}_{i+1} = P\overrightarrow{x_i}, \; i = 1, 2, 3, \ldots$

Two initial states $\overrightarrow{x}_0$: $\overrightarrow{e}_1$, $\overrightarrow{e}_2$



i	1	2	3	4	5	6	7
$\overrightarrow{e}_1$	$\begin{bmatrix} .8 \\ .2 \end{bmatrix}$	$\begin{bmatrix} .7 \\ .3 \end{bmatrix}$	$\begin{bmatrix} .65 \\ .35 \end{bmatrix}$	$\begin{bmatrix} .625 \\ .375 \end{bmatrix}$	$\begin{bmatrix} .6125 \\ .3875 \end{bmatrix}$	$\begin{bmatrix} .6062 \\ .3938 \end{bmatrix}$	$\begin{bmatrix} .6031 \\ .3969 \end{bmatrix}$
$\overrightarrow{e}_2$	$\begin{bmatrix} .3 \\ .7 \end{bmatrix}$	$\begin{bmatrix} .45 \\ .55 \end{bmatrix}$	$\begin{bmatrix} .525 \\ .475 \end{bmatrix}$	$\begin{bmatrix} .5625 \\ .4375 \end{bmatrix}$	$\begin{bmatrix} .5812 \\ .4188 \end{bmatrix}$	$\begin{bmatrix} .5906 \\ .4094 \end{bmatrix}$	$\begin{bmatrix} .5953 \\ .4047 \end{bmatrix}$

Apparently for both initial values the system runs towards $\begin{bmatrix} .6 \\ .4 \end{bmatrix}$.

16.1 Definition of Markov chains

Define

- A vector $\overrightarrow{v} = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}^T \in \mathbb{R}^n$ is called *probability vector* if $v_i > 0$ for $1 \leq i \leq n$ and $\sum_{i=1}^{n} v_i = 1$.
- A square matrix $A = \begin{bmatrix} \overrightarrow{v}_1 & \overrightarrow{v}_2 & \dots & \overrightarrow{v}_1 \end{bmatrix}$ with $\overrightarrow{v}_i$ probability vector for $1 \leq i \leq n$ is called *stochastic matrix*.

Markov chain

A *Markov chain* is a sequence of probability vectors $\overrightarrow{x}_1, \overrightarrow{x}_2, \dots$ with a stochastic matrix P such that

$$\overrightarrow{x}_{i+1} = P\overrightarrow{x}_i \qquad \forall i$$

We also call $\overrightarrow{x}_i$ *state vector*.

Questions connected to Markov chains

- What happens in the next (/after finitely many) steps?
- What is the long term behaviour of the system?

The long term behaviour of Markov chains

A stochastic matrix P is called **regular** if there is some $k \in \mathbb{N}$ such that P^k has only strictly positive entries.

Examples

$P = \begin{bmatrix} .5 & .25 & .25 \\ 0 & .25 & .25 \\ .5 & .5 & .5 \end{bmatrix}$ is regular since $P^2 = \begin{bmatrix} .375 & .3125 & .3125 \\ .125 & .1875 & .1875 \\ .5 & .5 & .5 \end{bmatrix}$

$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is not regular

The long term behaviour of Markov chains II

16.5 Theorem

If P is a regular $n \times n$ stochastic matrix, then P has a unique steady-state vector $\overrightarrow{q}$.
For any intial state $\overrightarrow{x}_0$ the Markov chain $\{x_k\}_{k\in\mathbb{N}_0}$ with $\overrightarrow{x}_{k+1} = P\overrightarrow{x}_k$ converges to $\overrightarrow{q}$ as $k \to \infty$.

To find a steady-state vector:

- Check if the stochastic matrix P is regular
- Compute as in 16.4.
  Do **not** try to compute $P^k\overrightarrow{x}_0$ for $k \to \infty$