TMA 4115 Matematikk 3

Lecture 21 for MTFYMA

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16.1 Definition of Markov chains

Define

- A vector $\overrightarrow{v} = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}^T \in \mathbb{R}^n$ is called *probability vector* if $v_i > 0$ for $1 \leq i \leq n$ and $\sum_{i=1}^n v_i = 1$.
- A square matrix $A = \begin{bmatrix} \overrightarrow{v}_1 & \overrightarrow{v}_2 & \dots & \overrightarrow{v}_1 \end{bmatrix}$ with $\overrightarrow{v}_i$ probability vector for $1 \leq i \leq n$ is called *stochastic matrix*.

Markov chain

A *Markov chain* is a sequence of probability vectors $\overrightarrow{x}_1, \overrightarrow{x}_2, \dots$ with a stochastic matrix P such that

$$\overrightarrow{x}_{i+1} = P\overrightarrow{x}_i \qquad \forall i$$

We also call $\overrightarrow{x}_i$ *state vector*.

Questions connected to Markov chains

- What happens in the next (/after finitely many) steps?
- What is the long term behaviour of the system?

In an example for two initial states the system converged towards a vector (in this context called **steady state vector**).

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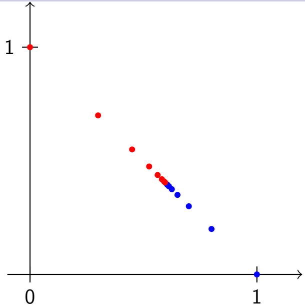
Example for a Markov chain

$$P = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix},$$

$\overrightarrow{x}_{i+1} = P\overrightarrow{x_i}, \; i = 1, 2, 3, \ldots$

Two initial states $\overrightarrow{x}_0$: $\overrightarrow{e}_1$, $\overrightarrow{e}_2$



i	1	2	3	4	5	6	7
$\overrightarrow{e}_1$	$\begin{bmatrix} .8 \\ .2 \end{bmatrix}$	$\begin{bmatrix} .7 \\ .3 \end{bmatrix}$	$\begin{bmatrix} .65 \\ .35 \end{bmatrix}$	$\begin{bmatrix} .625 \\ .375 \end{bmatrix}$	$\begin{bmatrix} .6125 \\ .3875 \end{bmatrix}$	$\begin{bmatrix} .6062 \\ .3938 \end{bmatrix}$	$\begin{bmatrix} .6031 \\ .3969 \end{bmatrix}$
$\overrightarrow{e}_2$	$\begin{bmatrix} .3 \\ .7 \end{bmatrix}$	$\begin{bmatrix} .45 \\ .55 \end{bmatrix}$	$\begin{bmatrix} .525 \\ .475 \end{bmatrix}$	$\begin{bmatrix} .5625 \\ .4375 \end{bmatrix}$	$\begin{bmatrix} .5812 \\ .4188 \end{bmatrix}$	$\begin{bmatrix} .5906 \\ .4094 \end{bmatrix}$	$\begin{bmatrix} .5953 \\ .4047 \end{bmatrix}$

In the long term the Markov chain converges towards the steady state vector $\begin{bmatrix} .6 \\ .4 \end{bmatrix}$.

The long term behaviour of Markov chains

A stochastic matrix P is called **regular** if there is some $k \in \mathbb{N}$ such that P^k has only strictly positive entries.

Examples

$$P = \begin{bmatrix} .5 & .25 & .25 \\ 0 & .25 & .25 \\ .5 & .5 & .5 \end{bmatrix} \text{ is regular since } P^2 = \begin{bmatrix} .375 & .3125 & .3125 \\ .125 & .1875 & .1875 \\ .5 & .5 & .5 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ is not regular}$$

The long term behaviour of Markov chains II

16.5 Theorem

If P is a regular $n \times n$ stochastic matrix, then P has a unique steady-state vector $\overrightarrow{q}$.
For any intial state $\overrightarrow{x}_0$ the Markov chain $\{x_k\}_{k \in \mathbb{N}_0}$ with $\overrightarrow{x}_{k+1} = P\overrightarrow{x}_k$ converges to $\overrightarrow{q}$ as $k \to \infty$.

To find a steady-state vector:

- Check if the stochastic matrix P is regular
- Compute as in 16.4.
  Do **not** try to compute $P^k\overrightarrow{x}_0$ for $k \to \infty$

Special vectors attached to matrices

The steady-state vectors which determine the long term behaviour of a Markov chain satisfy

$$P\overrightarrow{x} = \overrightarrow{x}$$

Idea: Study matrices using these special kind of vectors.

Vectors which "reproduce" via the following formula

$$A\overrightarrow{x} = \lambda\overrightarrow{x}, \quad \lambda \text{ a scalar}$$

will allow us to discover hidden structures in matrices.