TMA 4115 Matematikk 3

Lecture 22 for MTFYMA

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In today's lecture we will...

- discuss when two matrices are similar
- use similarity to relate matrices to diagonal matrices
- more on eigenvalues and eigenvectors

Eigenvector and eigenvalue

Definition of Eigenvectors and Eigenvalues

A a square matrix. $\overrightarrow{x}$ is **Eigenvector** (of A) with **eigenvalue** λ if

$$A\overrightarrow{x} = \lambda\overrightarrow{x} \quad \text{holds for } \lambda.$$

We note that...

- $\overrightarrow{v}$ and $\overrightarrow{w}$ eigenvectors for A (with eigenvalue λ), then $\overrightarrow{v} + r\overrightarrow{w}$ is eigenvector with eigenvalue λ
- A real matrix can have complex eigenvalues

We compute

- eigenvalues via the **characteristic polynomial** $\det(A - \lambda I)$.
- eigenvectors via $(A - \lambda I)\overrightarrow{x} = \overrightarrow{0}$ (after establishing that λ is eigenvalue!)

Example: Find eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

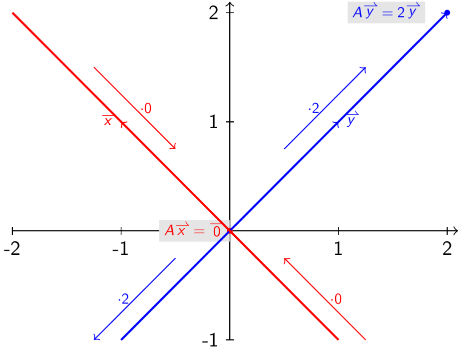
Calculate the roots of the **characteristic polynomial**

$$\det(A - \lambda I) = (1-\lambda)^2 - 1 = \lambda^2 - 2\lambda = (0-\lambda)(2-\lambda).$$

Roots of the characteristic polynomial = eigenvalues.
$\lambda = 0$ and $\lambda = 2$.

Now we also calculate eigenvectors. For $\lambda = 0$ and $\lambda = 2$ we use Gaussian elimination:
Find a non trivial solution of $(A - 0I)\overrightarrow{x} = \overrightarrow{0}$ and $(A - 2I)\overrightarrow{y} = \overrightarrow{0}$.
For example $\overrightarrow{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\overrightarrow{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Along the eigenvectors the matrix acts by scalar multiplication:



Observe that the eigenvectors even form a basis $\mathcal{B}$ for $\mathbb{R}^2$!

Observation: Matrix action in $\mathcal{B}$ coordinates

Recall $\mathcal{B}$-coordinates $[\overrightarrow{x}]_{\mathcal{B}}$, have invertible linear map ($\sim$ matrix) $P_{\mathcal{B}}$ with $P_{\mathcal{B}}[\overrightarrow{x}]_{\mathcal{B}} = \overrightarrow{x}$. Then

$$\begin{tikzcd}
\overrightarrow{x} \arrow[rr, "A\cdot"] \arrow[d, "P_{\mathcal{B}}^{-1}\cdot"] & & A\overrightarrow{x} \\
{[\overrightarrow{x}]_{\mathcal{B}}} \arrow[rr, "\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}\cdot"] & & {[A\overrightarrow{x}]_{\mathcal{B}}} \arrow[u, "P_{\mathcal{B}}\cdot"']
\end{tikzcd}$$

$\rightarrow$ A acts on the $\mathcal{B}$-coordinates as a diagonal matrix!

Note $A = P_{\mathcal{B}} \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} P_{\mathcal{B}}^{-1}$

Question:

Can we “transform” (all?) square matrices to diagonal matrices?

A a $n \times n$-matrix. Find a similar diagonal matrix (if possible).

1. Compute the eigenvalues of$A$,
2. Compute the eigenvectors associated to the eigenvalues. (Gaussian elimination!)
3. Check if we have n linear independent eigenvectors. If not: A is not diagonalisable!
   Hint: Eigenvectors of different eigenvalues are linearly independent.
4. Write eigenvectors as columns in a matrix P
5. Construct a diagonal matrix D whose diagonal entries are the eigenvalues corresponding to the columns in P
6. Then $A = PDP^{-1}$