TMA 4115 Matematikk 3

Lecture 25 for MTFYMA

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13. April 2016

In today's lecture we will ...

- learn more about orthogonality and orthogonal complements
- study orthogonal projection
- use the Gram-Schmidt algorithm to construct orthogonal (/orthonormal) bases

Inner product, length and orthogonality

Let $\overrightarrow{x}, \overrightarrow{y} \in \mathbb{R}^n$ with components x_i and y_i, respectively.

Dot product/ Inner product

$\overrightarrow{x} \cdot \overrightarrow{y} = \overrightarrow{x}^T \overrightarrow{y} = \sum_{i=1}^n x_i y_i$.

Can use this to define

- **Length** of a vector: $\|\overrightarrow{x}\| = \sqrt{\overrightarrow{x} \cdot \overrightarrow{x}}$,
- **Distance** between $\overrightarrow{x}$ and $\overrightarrow{y}$: $\text{dist}(\overrightarrow{u}, \overrightarrow{v}) = \|\overrightarrow{u} - \overrightarrow{v}\|$,
- **Orthogonality** of vectors: $\overrightarrow{x}$ and $\overrightarrow{y}$ are orthogonal to each other if and only if $\overrightarrow{x} \cdot \overrightarrow{y} = 0$.

Orthogonal Complement

Let $W \subseteq \mathbb{R}^n$ be non-empty. We say

- $\overrightarrow{z} \in \mathbb{R}^n$ is orthogonal to W if for all $\overrightarrow{v} \in W$, $\overrightarrow{v} \cdot \overrightarrow{z} = 0$
- $W^\perp$ is the set of all vectors in $\mathbb{R}^n$ orthogonal to W (**orthogonal complement** of W).

Example:

For $\overrightarrow{0} \in \mathbb{R}^n$ we have $\{\overrightarrow{0}\}^\perp = \mathbb{R}^n$ and $(\mathbb{R}^n)^\perp = \{\overrightarrow{0}\}$.

$$L = \text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\} \subseteq \mathbb{R}^2 \text{ then } L^\perp = \text{span}\left\{\begin{bmatrix}-1\\1\end{bmatrix}\right\}.$$

furthermore $\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}^\perp = \text{span}\left\{\begin{bmatrix}-1\\1\end{bmatrix}\right\} = L^\perp$

If P is the $x-y$-plane in $\mathbb{R}^3$ then $P^\perp = \text{span}\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

Orthogonal complement

20.14 Theorem

$W \subseteq \mathbb{R}^n$ a subspace with $W = \text{span}\{u_1, \ldots, u_p\}$.Then the following holds:

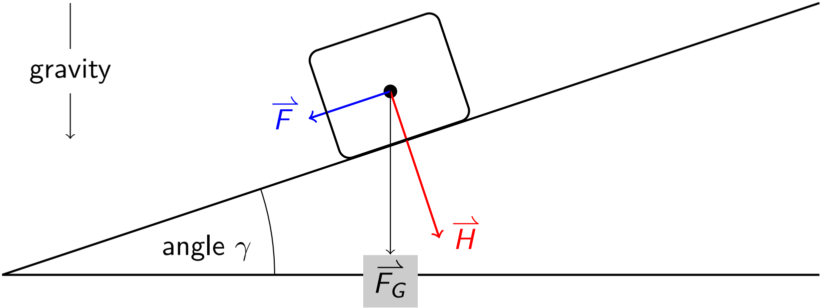
1. $\overrightarrow{x} \in W^\perp$ if and only if $\overrightarrow{x} \cdot u_i = 0$ for all $1 \leq i \leq p$
2. $W^\perp$ is a subspace of $\mathbb{R}^n$
3. $(W^\perp)^\perp = W$

Examples associated to a $m \times n$-matrix A

$(\text{Row}A)^\perp = \text{Nul}(A)$ and $(\text{Col}A)^\perp = \text{Nul}A^T$

Example: Splitting forces in physics

We know $\overrightarrow{F}_G$ and want to split the force into two orthogonal components:



Develop **orthogonal projections** to achieve this.

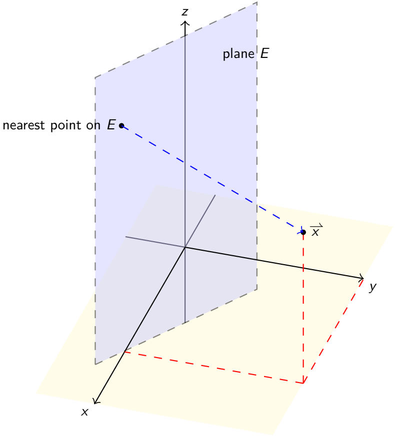
Nearest points on subspaces

The result of the orthogonal projection is the “nearest point” on a given subspace in $\mathbb{R}^n$.

Let $\overrightarrow{x} \in \mathbb{R}^3$,
$E \subseteq \mathbb{R}^n$ subspace

Nearest point on E:
The point $\overrightarrow{v}$ on E,
with minimal distance to $\overrightarrow{x}$

Observe: $\overrightarrow{x} - \overrightarrow{v}$ is perpendicular to E, i.e.
$\overrightarrow{x} - \overrightarrow{v} \in E^{\perp}$



21.7 The Gram-Schmidt Process

Let $W = \text{span}\{\overrightarrow{x}_1, \ldots, \overrightarrow{x}_p\} \subseteq \mathbb{R}^n$ and $\overrightarrow{x}_i \neq 0$ for $1 \leq i \leq p$.
Define

$$\begin{aligned}
\overrightarrow{u}_1 &= \overrightarrow{x}_1 \\
\overrightarrow{u}_2 &= \overrightarrow{x}_2 - \frac{\overrightarrow{x}_2 \cdot \overrightarrow{u}_1}{\overrightarrow{u}_1 \cdot \overrightarrow{u}_1}\overrightarrow{u}_1 \quad && \text{if } \overrightarrow{u}_2 = \overrightarrow{0} \text{ discard and construct} \\
& && \text{with same formula, now for } \overrightarrow{x}_3 \\
\vdots &= \vdots \qquad \vdots && \vdots \\
\overrightarrow{u}_p &= \overrightarrow{x}_p - \sum_{i=1}^{p-1} \frac{\overrightarrow{x}_p \cdot \overrightarrow{v}_i}{\overrightarrow{v}_i \cdot \overrightarrow{v}_i}\overrightarrow{v}_i && \text{if } \overrightarrow{u}_p = \overrightarrow{0} \text{ discard}
\end{aligned}$$

Then $\{\overrightarrow{u}_1, \ldots, \overrightarrow{u}_d\}$ is an orthogonal basis for W (for some $1 \leq d \leq p$) and

$$\text{span}\,\{\overrightarrow{u}_1, \ldots, \overrightarrow{u}_k\} = \text{span}\,\{\overrightarrow{x}_1, \ldots, \overrightarrow{x}_k\} \quad \text{for } 1 \leq k \leq d$$