TMA 4115 Matematikk 3

Lecture 3 for MTFYMA

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18. January 2014

Overview: Operations on complex numbers

For $w = a + ib = |w|(\cos(\theta) + i\sin(\theta))$,
$z = x + iy = |z|(\cos(\psi) + i\sin(\psi))$ we can compute...

Sum $w + z$

$w + z = (x + y) + i(b + y))$

Product $w \cdot z$ and powers z^n for $n \in \mathbb{N}$

$|w||z|(\cos(\theta + \psi) + i\sin(\theta + \psi))$ and $z^n = |z|^n(\cos(n\psi) + i\sin(n\psi))$

Conjugate $\overline{w}$

$\overline{w} = a - ib$

Reciprocal z^{-1} and quotient $\frac{w}{z}$ for $z \neq 0$

$z^{-1} = \frac{\overline{z}}{|z|^2}$ and $\frac{w}{z} = \frac{w \cdot \overline{z}}{|z|^2}$

Roots of complex numbers

Aim: For a given complex number z find all complex numbers w (roots) which solve

$$w^2 = z$$

Or more generally for $n \in \mathbb{N}$ the numbers w which solve

$$w^n = z \tag{1}$$

these are called *nth roots*.

From 1.8 in last weeks lecture: If w is an nth root of z, then by (1)

$$|w|^n = |z| \quad \left(\text{or } |w| = \sqrt[n]{|z|}\right) \text{ and}$$

$$n \cdot \arg(w) = \arg(z) \quad \left(\text{or } \arg(w) = \frac{\arg(z)}{n}\right)$$

nth roots of complex numbers

Let $z = r(\cos(\theta) + i\sin(\theta))$ be a complex number and $n \in \mathbb{N}$.

We call

$$z_1 = \sqrt[n]{r}\left(\cos\left(\frac{\theta}{n}\right) + i\sin\left(\frac{\theta}{n}\right)\right)$$

the *principal nth root of z*.

Examples:

$z = i = 1(\cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2}))$ the principal nth root
is $\cos(\frac{\pi}{2n}) + i\sin(\frac{\pi}{2n})$

$z = 0$ the principal nth root is 0.

Roots for $z = r(\cos(\theta) + i\sin(\theta)) \neq 0$ and $n \in \mathbb{N}$

There are n distinct nth roots which can be computed via

$$
\begin{aligned}
z_1 &= \sqrt[n]{r}\left(\cos\left(\frac{\theta}{n}\right) + i\sin\left(\frac{\theta}{n}\right)\right)\\
z_2 &= \sqrt[n]{r}\left(\cos\left(\frac{\theta+2\pi}{n}\right) + i\sin\left(\frac{\theta+2\pi}{n}\right)\right)\\
z_3 &= \sqrt[n]{r}\left(\cos\left(\frac{\theta+4\pi}{n}\right) + i\sin\left(\frac{\theta+4\pi}{n}\right)\right)\\
\vdots \quad & \quad \vdots\\
z_n &= \sqrt[n]{r}\left(\cos\left(\frac{\theta+2(n-1)\pi}{n}\right) + i\sin\left(\frac{\theta+2(n-1)\pi}{n}\right)\right)
\end{aligned}
$$

Complex functions and the exponential map

Our next goal is to define functions depending on complex variables.

We need these functions to solve differential equations.

How do we define define a *complex function*?
- Copy the definition of a real function and use complex variables!

Real functions

Recall that a *real function* is a triple

$$f\colon U \to V, \quad x \mapsto f(x)$$

where the parts of the triple are

$U \subseteq \mathbb{R}$ the *domain* (i.e. the numbers we apply f to)

$V \subseteq \mathbb{R}$ the *codomain* (must contain all values of f)

$x \mapsto f(x)$ a *rule* assigning to each x an element $f(x)$

Examples:

$$g\colon (0,1) \to (0,\infty), \quad x \mapsto \frac{1}{x}$$
$$\sin\colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto \sin(x)$$

Here (a, b) is the open interval from a to b.