TMA 4115 Matematikk 3

Lecture 4 for MTFYMA

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Complex functions

A *complex function* is a triple

$$f\colon U \to V, \quad x \mapsto f(x)$$

where $U \subseteq \mathbb{C}$ (domain), $V \subseteq \mathbb{C}$ (codomain)
and $x \mapsto f(x)$ is a rule which assigns to $x \in U$ a unique $f(x) \in V$.

Example: Complex polynomials

A function $p\colon \mathbb{C} \to \mathbb{C}, \quad p(z) = \sum_{k=0}^{n} a_k z^k$ for $a_0, \ldots, a_n \in \mathbb{C}$ and $n \in \mathbb{N}$ is called *(complex) polynomial*.

For a complex polynomial p a number z_0 is called *root* if $p(z_0) = 0$.

Quadratic formula for roots

For $p(z) = az^2 + bz + c, a \neq 0$ the roots of p are given by

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad (\star)$$

Proof:

$$\begin{aligned} 0 &= p(z) = az^2 + bz + c \\ \Leftrightarrow 0 &= 4a^2z^2 + 4abz + 4ac \\ &= (2az)^2 + 4abz + b^2 - b^2 + 4ac \\ \Leftrightarrow b^2 - 4ac &= (2az + b)^2 \\ \Leftrightarrow \pm\sqrt{b^2 - 4ac} &= 2az + b \end{aligned}$$

which yields $(\star)$ after solving for z. □

We already know how to solve the differential equations

$$y'(t) = \frac{d}{dt}y(t) = f(y, t)$$

if the function f is "nice".

We call such an equation a *first order differential equation*, because it **only involves the first derivative** of the unknown y.

First order differential equations are "simple", but the (physical) world is complicated, hence it is described by more complicated differential equations.

3.1 Newtons second law

Newtons second law

The *acceleration* a of a body with *mass* m is proportional to the *net force* F via

$$F = ma \tag{1}$$

Question: What is the displacement $y(t)$ of the body from a reference point?

Acceleration a is rate of change of velocity v, i.e.

$$a = \frac{d}{dt}v = v' \tag{2}$$

Velocity v is rate of change of the displacement, i.e.

$$v = \frac{d}{dt}y = y' \tag{3}$$

Thus (2) becomes $a = y''$.

What is the displacement $y(t)$ of the body from a reference point?

Finally, the net force F depends on time, displacement and velocity, thus F is a function

$$F(t, y, v) = F(t, y, y') \quad \text{(using (3))}$$

With Newtons second law (1) obtain

$$F(t, y, y') = my''$$

as a differential equation for the displacement $y(t)$.
This is a **second order differential equation** since it involves derivatives of y of up to second order.

3.2 Definition

A *second order differential equation* is an equation of the form

$$\frac{d^2}{dt^2}y(t) = f\left(y, \frac{d}{dt}y, t\right) \tag{4}$$

where f is a given function.

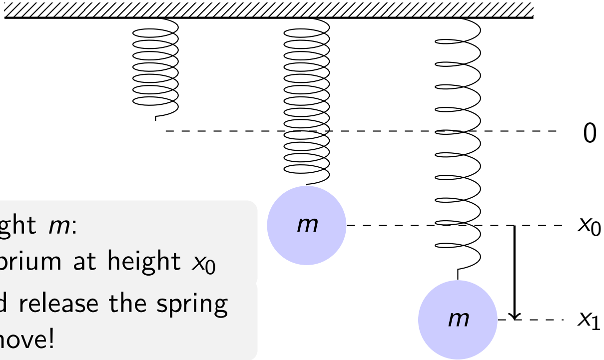
A *solution* to the second order equation (4) is a function y which is twice continuously differentiable and satisfies (4).

Usually we write $y' = \frac{d}{dt}y(t)$ and $y'' = \frac{d^2}{dt^2}y(t)$, thus (4) reads:

$$y'' = f(t, y, y') \tag{5}$$

3.3 The vibrating spring

We consider a spring suspended from a beam:



Attach weight m:
New equilibrium at height x_0

Stretch and release the spring
...it will move!

Forces acting on weight in motion:

damping force $D(v)$ (depends on velocity $v = x'$),
external force $F(t)$, **Restoring force** $R(x)$ and **gravity** mg.

A model for the position $x(t)$ of the spring

Newtons second law (1) for the spring

$$\begin{aligned} ma &= \text{total force acting on the weight} \\ &= R(x) + mg + D(v) + F(t) \end{aligned} \tag{6}$$

Velocity $v = x'$ and acceleration $a = v' = x''$, thus (6) becomes

$$mx'' = R(x) + mg + D(x') + F(t). \tag{7}$$

Hooke's law

For some springs, experiments show that the restoring force is $R(x) = -kx$ for $k > 0$ constant and small x.

Assuming Hooke's law, (7) becomes

$$mx'' = -kx + mg + D(x') + F(t). \tag{8}$$