TMA 4115 Matematikk 3

Lecture 5 for MTFYMA

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3.2 Definition

A *second order differential equation* is an equation of the form

$$\frac{d^2}{dt^2}y(t) = f\left(y, \frac{d}{dt}y, t\right) \tag{4}$$

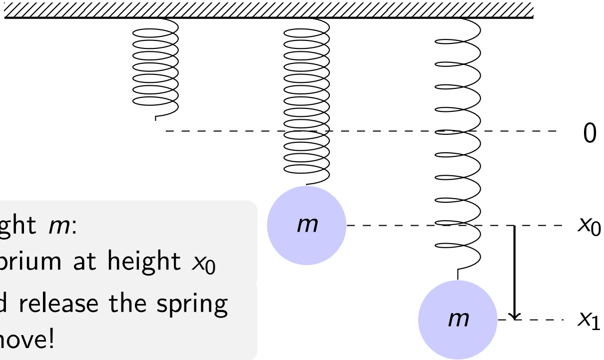
where f is a given function. Usually we write instead of (4)

$$y'' = f(t, y, y'). \tag{5}$$

A *solution* to the second order equation (4) is a function y which is twice continuously differentiable and satisfies (4).

3.3 The vibrating spring

We consider a spring suspended from a beam:



Attach weight m:
New equilibrium at height x_0

Stretch and release the spring
. . . it will move!

Forces acting on weight in motion:

damping force $D(v)$ (depends on velocity $v = x'$),
external force $F(t)$, **Restoring force** $R(x)$ and **gravity** mg.

A model for the position $x(t)$ of the spring

Newtons second law shows

$$\begin{aligned} ma &= \text{total force acting on the weight} \\ &= R(x) + mg + D(v) + F(t) \end{aligned} \tag{6}$$

Velocity $v = x'$, acceleration $a = v' = x''$ and Hooke's law let us rewrite (6) as

$$mx'' = -kx + mg + D(x') + F(t) \qquad k > 0. \tag{7}$$

More experimental results transform (7) into

$$mx'' = -kx + mg - \mu x' + F(t) \qquad \mu, k > 0. \tag{8}$$

This equation measures the position x.

A model for the displacement $y(t)$ of the spring

Instead of (8) we want an equation for $y = x - x_0$.
Assume: $F(t) = 0$ and $x = x_0$ then $x' = x'' = 0$ and (8) reads

$$0 = -kx_0 + mg \qquad \Leftrightarrow \qquad kx_0 = mg$$

Substituting $y = x - x_0$ and $mg = kx_0$ into (8) we obtain

$$my'' = -ky - \mu y' + F(t) \tag{9}$$

or equivalent

$$my'' + \mu y' + ky = F(t)$$

Is there a solution for every 2nd order differential equation?

3.6 Theorem

Let p, q and g be continuous functions with domain (α, β) (open interval for $\alpha < \beta$). Fix $t_0 \in (\alpha, \beta)$ and $y_0, y_1 \in \mathbb{R}$. There is one and only one function $y\colon (\alpha, \beta) \to \mathbb{R}$ which solves

$$(\star) \begin{cases} y'' + p(t)y' + q(t)y = g(t), & \text{for } t \in (\alpha, \beta) \\ y(t_0) = y_0, y'(t_0) = y_1 \end{cases}$$

3.7 Remark: Solution of $(\star)$ exists on **all** of (α, β)

We need $y(t_0) = y_0, y'(t_0) = y_1$ to get a unique solution.
(compare with chapter on complex exponential function)