TMA 4115 Matematikk 3

Lecture 6 for MTFYMA

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Linear (second order) differential equations are of the form

$$y'' + p(t)y' + q(t)y = g(t)$$

where p, q and g depend only on t (not on y).

If the forcing term g is 0, the equation is **homogeneous**, otherwise **inhomogeneous**.

An **initial value problem** (IVP) is a differential equation with enough initial values to specify a solution.

$u, v\colon (\alpha,\beta) \to \mathbb{R}$ are **linearly independent** (on (α,β)) if there is no $C \in \mathbb{C}$ with $u(t) = Cv(t)$ for all $t \in (\alpha,\beta)$.

Tests for linear independence:

- **Inspection**
- if u, v solve the same linear homogeneous differential equation use their **Wronskian** $W(t) = u(t)v'(t) - v(t)u'(t)$. If $W(t) \neq 0$ then u, v are linearly independent.

Structure of the general solution

3.14 Theorem

Let y_1, y_2 be linearly independent solutions to

$$y'' + p(t)y' + q(t)y = 0$$

Then the **general solution** to the differential equation is

$$y(t) = Ay_1(t) + By_2(t)$$

where $A, B \in \mathbb{C}$.

We call two linearly independent solutions for a second order homogeneous linear equation a **fundamental set** of solutions.

Strategy to solve homogeneous linear differential equations

$$y'' + p(t)y' + q(t)y = 0$$

Obtain general solution

- Find two solutions u, v
- Check that u, v are linearly independent (Wronskian!)
- General solution $Au + Bv$

Solve IVP $y'' + p(t)y' + q(t)y = 0, y(t_0) = y_0, y'(t_0) = y_1$

- Need general solution to $y'' + p(t)y' + q(t)y = 0$
- Use $y(t_0) = y_0$ and $y'(t_0) = y_1$ to determine A and B

3.16 Example

We know that for $\omega \neq 0$, $\sin(\omega t)$ and $\cos(\omega t)$ solve

$$y'' + \omega^2 y = 0 \tag{12}$$

Compute Wronskian $W(t) = \omega(\cos(\omega t)^2 + \sin(\omega t)^2) = \omega \neq 0$.
$\{\sin(\omega t), \cos(\omega t)\}$ is a fundamental set of solutions and

$$y(t) = A\sin(\omega t) + B\cos(\omega t) \;\; A, B \in \mathbb{C} \text{ (general solution)}$$

Now initial conditions $y(0) = 2$ and $y'(0) = 1$: Insert the general solution:

$$\begin{aligned} 2 &= y(0) = A\sin(0) + B\cos(0) = B \\ 1 &= y'(0) = A\omega\cos(0) - B\omega\sin(0) = A\omega \end{aligned}$$

$y(t) = \frac{1}{\omega}\sin(\omega t) + 2\sin(\omega t)$ solves IVP (12), $y(0) = 2$, $y'(0) = 1$.

Problem: How to find any solution?

We know what to do if we already found solutions to a linear homogeneous equation. However, how do we find these solutions?

Goal: Construct solutions for simpler homogeneous linear equations, i.e. equations with constant coefficients.

$$(\text{i.e. } p \equiv \text{const}, q \equiv \text{const})$$

Idea: Consider

$$y' + qy = 0, \quad q \in \mathbb{C}$$

We know that $y(t) = Ce^{-qt}$ solves the equation for all $C \in \mathbb{C}$. Try $y(t)$ as a solution to a second order homogeneous equation.