TMA 4115 Matematikk 3

Lecture 6 Part 2 for MTFYMA

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A linear (second order) differential equation

$$y'' + py' + qy = g(t) \tag{1}$$

where $p, q \in \mathbb{R}$ and g is a given function is called linear (inhomogeneous) differential equation with constant coefficients.

Solve the homogeneous equation associated to (1):

$$y'' + py' + qy = 0 \tag{4}$$

using the associated **characteristic polynomial**

$$\lambda^2 + p\lambda + q = 0. \tag{5}$$

We distinguish three cases depending on the **characteristic roots**:

Solutions for equations with constant coefficients

Case 1 $p^2 - 4q > 0$, i.e. two distinct, real roots λ_1 and λ_2. Fundamental set of solutions:

$$y_1(t) = e^{\lambda_1 t} \quad y_2(t) = e^{\lambda_2 t}$$

Case 2 $p^2 - 4q < 0$, we have two distinct, complex roots $\lambda_1 = a + ib$ and $\overline{\lambda_1}$. Fundamental set of (real valued) solutions:

$$y_1(t) = e^{at}\cos(bt) \quad y_2(t) = e^{at}\sin(bt)$$

Case 3 $p^2 - 4q = 0$, there is one repeated real root λ_1. Fundamental set of solutions:

$$y_1(t) = e^{\lambda_1 t}, \quad y_2(t) = te^{\lambda_1 t}$$

Harmonic motion

An important example of a linear equations with constant coefficients is the equation of **harmonic motion**

$$y'' + 2cy' + \omega_0^2 y = f(t) \qquad (11)$$

with $c, \omega_0 \in \mathbb{R}$ and $f(t)$ a given function.We call

c, the dampening parameter

ω_0, the natural frequency

Example: The spring equation (Chapter 3)

$$my'' = -ky - \mu y' + F(t) \qquad k, \mu > 0$$

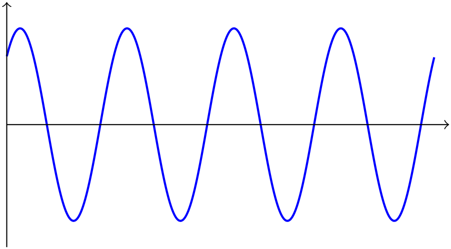
or equivalently

$$y'' + \frac{\mu}{m} y' + \frac{k}{m} y = \frac{F(t)}{m} \qquad (12)$$

Simple Harmonic motion (i.e. $\mu = 0 = F(t)$)

Have $\lambda^2 + \frac{k}{m} = 0$ with (two complex) characteristic roots $\pm\sqrt{-\frac{k}{m}} = \pm i\sqrt{\frac{k}{m}} = \pm i\omega_0$. General (real) solution of (12) is

$$\begin{aligned} y(t) &= Ae^{0t}\cos(\omega_0 t) + Be^{0t}\sin(\omega_0 t) \\ &= A\cos(\omega_0 t) + B\sin(\omega_0 t) \qquad A, B \in \mathbb{R} \end{aligned}$$



plotted for $A, B = 1$, $\omega_0 = 4$

No Damping or forcing: Solution oscillates with natural frequency $\omega_0 = \sqrt{\frac{k}{m}}$