TMA4115 Calculus 3
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4.1 Vector Spaces and Subspaces

1. Let V be the first quadrant in the xy-plane; that is, let

$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, y \geq 0 \right\}$$

a. If $\mathbf{u}$ and $\mathbf{v}$ are in V, is $\mathbf{u} + \mathbf{v}$ in V? Why?

b. Find a specific vector $\mathbf{u}$ in V and a specific scalar c such that $c\mathbf{u}$ in *not* in V. (This is enough to show that V is *not* a vector space).

9. Let H be the set of all vectors of the form $\begin{bmatrix} s \\ 3s \\ 2s \end{bmatrix}$. Find a vector $\mathbf{v}$ in $\mathbb{R}^3$ such that $H = \operatorname{span}\{\mathbf{v}\}$. Why does this show that H is a subspace of $\mathbb{R}^3$?

11. Let W be the set of all vectors of the form $\begin{bmatrix} 5b + 2c \\ b \\ c \end{bmatrix}$, where b and c are arbitrary. Find vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ such that $W = \operatorname{span}\{\mathbf{u}, \mathbf{v}\}$. Why does this show that W is a subspace of $\mathbb{R}^3$?

32. (Optional Extra) Let H and K be subspaces of a vector space V. The **intersection** of H and K, written as $H \cap K$, is the set of $\mathbf{v}$ in V that belong to both H and K. Show that $H \cap K$ is a subspace of V. (See the figure on page 215 in the textbook). Give an example in $\mathbb{R}^2$ to show that the union of two subspaces is not, in general, a subspace.

4.2 Null Spaces, Column Spaces, and Linear Transformations

6. Find an explicit description of Nul$A$ by listing vectors that span the null space.

$$A = \begin{bmatrix} 1 & 5 & -4 & -3 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

21. With A as in Exercise 17, find a nonzero vector in Nul$A$ and a nonzero vector in Col$A$.

$$A = \begin{bmatrix} 2 & -6 \\ -1 & 3 \\ -4 & 12 \\ 3 & -9 \end{bmatrix}$$

24. Let $A = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$. Determine if $\mathbf{w}$ is in Col$A$. Is $\mathbf{w}$ in Nul$A$?

28. Consider the following two systems of equations:

$$\begin{aligned} 5x_1 + x_2 - 3x_3 &= 0 \\ -9x_1 + 2x_2 + 5x_3 &= 1 \\ 4x_1 + x_2 - 6x_3 &= 9 \end{aligned} \qquad \begin{aligned} 5x_1 + x_2 - 3x_3 &= 0 \\ -9x_1 + 2x_2 + 5x_3 &= 5 \\ 4x_1 + x_2 - 6x_3 &= 45 \end{aligned}$$

It can be shown that the first system has a solution. Use this fact and the theory form this section to explain why the second system must also have a solution. (Make no row operations).

32. (Optional Extra) Define a linear transformation $T : \mathbb{P}_2 \to \mathbb{R}^2$ by $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(0) \end{bmatrix}$. Find polynomials in $\mathbf{p}_1$ and $\mathbf{p}_2$ in $\mathbb{P}_2$ that span the kernel of T, and describe the range of T.

4.3 Linearly Independent Sets; Bases

Determine which sets in Exercises **1**, **3**, and **8** are bases for $\mathbb{R}^3$. Of the sets that are *not* bases, determine which ones are linearly independent and which ones span $\mathbb{R}^3$. Justify your answers.

1. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

3. $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix}$

8. $\begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}$

11. Find a basis for the set of vectors on $\mathbb{R}^3$ in the plane $x + 2y + z = 0$. (*Hint:* Think of the equation as a "system" of homogeneous equations).

23. Suppose $\mathbb{R}^4 = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_4\}$. Explain why $\{\mathbf{v}_1, \ldots, \mathbf{v}_4\}$ is a basis for $\mathbb{R}^4$.

Exercise **29** and **30** show that every basis for $\mathbb{R}^n$ must contain exactly n vectors.
29. Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a set of k vectors in $\mathbb{R}^n$, with $k < n$. Use a theorem from section 1.4 to explain why S cannot be a basis for $\mathbb{R}^n$.

30. Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a set of k vectors in $\mathbb{R}^n$, with $k > n$. Use a theorem from Chapter 1 to explain why S cannot be a basis for $\mathbb{R}^n$.

32. (Optional Extra) This exercise reveal an important connection between linear independence and linear transformations and provide practice using the definition of linear dependence. Let V and W be vectors spaces, let $T : V \to W$ be a linear transformation, and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a subset of V.
Suppose that T is a one-to-one transformation, so that an equation $T(\mathbf{u}) = T(\mathbf{v})$ always implies $\mathbf{u} = \mathbf{v}$. Show that if the set of images $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_p)\}$ is linearly dependent, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent. This fact shows that *a one-to-one linear transformation maps a linearly independent set into a linearly independent set* (because in this case the set of images cannot be linearly dependent.)

Extra (Optional Extra) Prove the following statements:

1) if a $m \times n$-matrix A has a left-inverse (i.e., if there is a C such that $CA = I$), then its columns are independent. (*Hint:* assume that there is a vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$, and show that $\mathbf{x} = \mathbf{0}$). Convince yourself that it does indeed prove statement 1).

2) if a $m \times n$-matrix A has a right-inverse (i.e., if there is C such that $AC = I$), then its columns span $\mathbb{R}^m$. (*Hint:* show that the equation $AC\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$. Deduce that $A\mathbf{y} = \mathbf{b}$ has a solution for all $\mathbf{b}$. Convince yourself that it does indeed prove statement 2).