Norges teknisk–naturvitenskapelige universitet
Institutt for matematiske fag

TMA4115 Calculus 3
Spring 2016

Øving 10

4.4 Coordinate Systems

In Exercise **3** and **4**, find the vector $\mathbf{x}$ determined by the given coordinate vector $[\mathbf{x}]_\mathcal{B}$ and the given basis $\mathcal{B}$.

3.

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ -7 \\ 0 \end{bmatrix} \right\}, \quad [\mathbf{x}]_\mathcal{B} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$$

4.

$$\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -7 \\ 3 \end{bmatrix} \right\}, \quad [\mathbf{x}]_\mathcal{B} = \begin{bmatrix} -4 \\ 8 \\ -7 \end{bmatrix}$$

7. Find the coordinate vector $[\mathbf{x}]_\mathcal{B}$ for $\mathbf{x}$ relative to the given basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$.

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -3 \\ 4 \\ 9 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 8 \\ -9 \\ 6 \end{bmatrix}$$

21. (Optional Extra) Let $\mathbf{B} = \left\{ \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ 9 \end{bmatrix} \right\}$. Since the coordinate mapping determined by $\mathcal{B}$ is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^2$, this mapping must be implemented by some 2×2 matrix A. Find it. (*Hint:* Multiplication by A should transform a vector $\mathbf{x}$ into its coordinate vector $[\mathbf{x}]_\mathcal{B}$.)

4.5 The Dimension of a Vector Space

3. For each subspace in the exercise, (a) find a basis, and (b) state the dimension.

$$\left\{ \begin{bmatrix} 2c \\ a-b \\ b-3c \\ a+2b \end{bmatrix} : a, b, c \in R \right\}$$

9. Find the dimension of the subspace of all vectors in R^3 whose first and third entries are equal.

12. Find the dimension of the subspace spanned by the given vectors.

$$\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -8 \\ 6 \\ 5 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \\ 7 \end{bmatrix}$$

13. Determine the dimensions of $Nul\ A$ and $Col\ A$.

$$A = \begin{bmatrix} 1 & -6 & 9 & 0 & -2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

21. The first four Hermite polynomials are 1, $2t$, $-2+4t^2$, and $-12t+8t^3$. These polynomials arise naturally in the study of certain important differential equations in mathematical physics. Show that the first four Hermite polynomials form a basis for $\mathbb{P}_3$.

26. (Optional Extra) Let H be a n-dimensional subspace of an n-dimensional space V. Show that $H = V$.

4.6 Rank

3. Assume that the matrix A is row equivalent to B. Without calculations, list rank$A$ and dimNul$A$. Then find bases for Col$A$, Row$A$, and Nul$A$.

$$A = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

5. If a 3×8 matrix A has rank 3, find dim $Nul\ A$, dim $Row\ A$, and rank A^T.

13. If A is a 7×5 matrix, what is the largest possible rank of A? If A is a 5×7 matrix, what is the largest possible rank of A? Explain your answers.

14. If A is a 4×3 matrix, what is the largest possible dimension of the row space of A? If A is a 3×4 matrix, what is the largest possible dimension of the row space of A? Explain.

17. A is an $m \times n$ matrix. Mark each statement True or False. Justify each answer.

a. The row space of A is the same as the column space of A^T.

b. If B is any echelon form of A, and if B has three nonzero rows,then the first three rows of A form a basis for $RowA$.

c. The dimensions of the row space and the column space of A are the same, even if A is not square.

d. The sum of the dimensions of the row space and the null space of A equals the number of rows in A.

e. On a computer, row operations can change the apparent rank of a matrix.

28. (Optional Extra) This Exercise concern an $m \times n$ matrix A and what are often called the *fundamental subspaces* determined by A. Justify the following equalities:

a. $\dim \text{Row}A + \dim \text{Nul}A = n$ *Number of columns of A*

b. $\dim \text{Col}A + \dim \text{Nul}A^T = m$ *Number of rows of A*

4.9 Applications to Markov Chains

3. On any given day, a student is either healthy or ill. Of the students who are healthy today, 95% will be healthy tomorrow. Of the students who are ill today, 55% will still be ill tomorrow.

a. What is the stochastic matrix for this situation?

b. Suppose 20% of the students are ill on Monday. What fraction or percentage of the students are likely to be ill on Tuesday? On Wednesday?

c. If a student is well today, what is the probability that he or she will be well two days from now?

7. Find the steady-state vector.

$$\begin{bmatrix} .7 & .1 & .1 \\ .2 & .8 & .2 \\ .1 & .1 & .7 \end{bmatrix}$$

18. (Optional Extra) Show that every 2×2 stochastic matrix has at least one steady-state vector. Any such matrix can be written in the form $P = \begin{bmatrix} 1-\alpha & \beta \\ \alpha & 1-\beta \end{bmatrix}$, where α and β are constants between 0 and 1. (There are two linearly independent steady-state vectors if $\alpha = \beta = 0$. Otherwise, there is only one.)