Norges teknisk–naturvitenskapelige universitet
Institutt for matematiske fag

5.1 Eigenvectors and Eigenvalues

6. Is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}$? If so, find the eigenvalue.

7. Is $\lambda = 4$ an eigenvalue of $\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}$? If so, find one corresponding eigenvector.

9. Find a basis for the eigenspace corresponding to each listed eigenvalue.

$$A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}, \quad \lambda = 1,\, 5$$

21. A is an $n \times n$ matrix. Mark each statement True of False. Justify each answer.

a. If $A\mathbf{x} = \lambda\mathbf{x}$ for some vector $\mathbf{x}$, then λ is an eigenvalue of A.

b. A matrix A is not invertible if and only if 0 is an eigenvalue of A.

c. A number c is an eigenvalue of A if and only if the equation $(A - cI)\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

d. Finding an eigenvector of A may be difficult, but checking whether a given vector is in fact an eigenvector is easy.

e. To find the eigenvalue of A, reduce A to echelon form.

23. Explain why a 2×2 matrix can have at most two distinct eigenvalues. Explain why an $n \times n$ matrix can have at most n distinct eigenvalues.

In Exercises **31** and **32**, let A be the matrix of the linear transformation T. Without writing A, find an eigenvalue of A and describe the eigenspace.
31. T is the transformation on $\mathbb{R}^2$ that reflects points across some line through the origin.

32. T is the transformation on $\mathbb{R}^3$ that rotates points about some line through the origin.

5.2 The Characteristic Equation

Find the characteristic polynomial and the eigenvalues of the matrices in Exercise **1** and **5**

1.

$$\begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$$

5.

$$\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$$

11. This Exercise require techniques from Section 3.1. *Find the characteristic polynomial of the matrix* using either a cofactor expansion or the special formula for 3×3 determinants described prior to Exercises 15-18 in Section 3.1 (*Note:* Finding the characteristic polynomial of a 3×3 matrix is not easy to do with just row operations, because the variable λ is involved.)

$$\begin{bmatrix} 4 & 0 & 0 \\ 5 & 3 & 2 \\ -2 & 0 & 2 \end{bmatrix}$$

15. List the eigenvalues, repeated according to their multiplicities.

$$\begin{bmatrix} 4 & -7 & 0 & 2 \\ 0 & 3 & -4 & 6 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

24. Show that if A and B are similar, then $\det A = \det B$.

25. Let $A = \begin{bmatrix} .6 & .3 \\ .4 & .7 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}, \mathbf{x}_0 = \begin{bmatrix} .5 \\ .5 \end{bmatrix}$. (*Note:* A is the stochastic matrix studied in Example 5 of section 4.9.)

a. Find a basis for $\mathbb{R}^2$ consisting of $\mathbf{v}_1$ and another eigenvector $\mathbf{v}_2$ of A.

b. Verify that $\mathbf{x}_0$ may be written in the form $\mathbf{x}_0 = \mathbf{v}_1 + c\mathbf{v}_2$.

c. For $k = 1, 2, \dots$, define $\mathbf{x}_k = A^k\mathbf{x}_0$. Compute $\mathbf{x}_1$ and $\mathbf{x}_2$, and write a formula for $\mathbf{x}_k$. Then show that $\mathbf{x}_k \to \mathbf{v}_1$ as k increases.

18. (Optional Extra) It can be shown that the algebraic multiplicity of an eigenvalue λ is always greater of equal to the dimension of the eigenspace corresponding to λ. Find h in the matrix A below such that the eigenspace for $\lambda = 5$ is two-dimensional:

$$\begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & h & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

19. (Optional Extra) Let A be an $n \times n$ matrix, and suppose A has n real eigenvalues, $\lambda_1, \dots, \lambda_n$, repeated according to multiplicities, so that

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda).$$

Explain why $\det A$ is the product of the n eigenvalues of A. (This result is true for any square matrix when complex eigenvalues are considered.)

27. (Optional Extra) Let

$$A = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} .3 \\ .6 \\ .1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \text{ and } \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

a. Show that $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are eigenvectors of A. (*Note:* A is the stochastic matrix studied in Example 3 of Section 4.9.)

b. Let $\mathbf{x}_0$ be any vector in $\mathbb{R}^3$ with nonnegative entries whose sum is 1. (In Section 4.9, $\mathbf{x}_0$ was called a probability vector.) Explain Why there are constants c_1, c_2, and c_3 such that $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$. Compute $\mathbf{w}^T\mathbf{x}_0$, and deduce that $c_1 = 1$.

c. For $k = 1, 2, \dots$, define $\mathbf{x}_k = A^k\mathbf{x}_0$, with $\mathbf{x}_0$ as in part (b). Show that $\mathbf{x}_k \to \mathbf{v}_1$ as k increases.

5.3 Diagonalization

1. Let $A = PDP^{-1}$ and compute A^4.

$$P = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

5. The matrix A is factored in the form PDP^{-1}. Use the Diagonalization Theorem to find the eigenvalues of A and a basis for each eigenspace.

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/4 & 1/2 & -3/4 \\ 1/4 & -1/2 & 1/4 \end{bmatrix}$$

Diagonalize the matrices in Exercise **7** and **12**, if possible.
7.

$$\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$$

12. The eigenvalues for the matrix are $\lambda = 2$, 8

$$\begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

31. (Optional Extra) Construct a nonzero 2×2 matrix that is invertible but not diagonalizable.

32. (Optional Extra) Construct a nondiagonal 2×2 matrix that is diagonalizable, but not invertible.

5.5 Complex Eigenvalues

1. Let the $\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$ act on $\mathbb{C}^2$. Find the eigenvalues and a basis for each eigenspace in $\mathbb{C}^2$.

7. Use Example 6 to list the eigenvalues of A. In each case, the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is the composition of a rotation and a scaling. Give the angle ϕ of the rotation, where $-\pi \leq \phi \leq \pi$, and give the scale factor r.

$$A = \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$$

13. Find an invertible matrix P and a matrix C on the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that the given matrix has the form $A = PCP^{-1}$. Use the information from Exercise **1**.

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$$

Chapter 7 will focus on matrices A with the property that $A^T = A$. Exercises **23** and **24** show that every eigenvalue of such a matrix is necessarily real.

23. (Optional Extra) Let A be an $n \times n$ real matrix with the property that $A^T = A$, let $\mathbf{x}$ be any vector in $\mathbb{C}^n$, and let $q = \bar{\mathbf{x}}^T A\mathbf{x}$. The equalities below show that q is a real number by verifying that $\bar{q} = q$. Give a reason for each step.

$$\bar{q} = \overline{\bar{\mathbf{x}}^T A\mathbf{x}} \overset{(a)}{=} \mathbf{x}^T \overline{A\mathbf{x}} \overset{(b)}{=} \mathbf{x}^T A\bar{\mathbf{x}} \overset{(c)}{=} (\mathbf{x}^T A\bar{\mathbf{x}})^T \overset{(d)}{=} \bar{\mathbf{x}}^T A^T \mathbf{x} \overset{(e)}{=} q$$

24. (Optional Extra) Let A be an $n \times n$ real matrix with the property that $A^T = A$. Show that if $A\mathbf{x} = \lambda\mathbf{x}$ for some nonzero vector $\mathbf{x}$ in $\mathbb{C}^n$, then, in fact, λ is real and the real part of $\mathbf{x}$ is an eigenvector of A. (*Hint:* Compute $\bar{\mathbf{x}}^T A\mathbf{x}$, and use Exercise **23**. Also, examine the real and imaginary parts of $A\mathbf{x}$.)