# TMA 4115 Matematikk 3

## Lecture for MTFYMA

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Topics in todays lecture...

- Markov chains

# Two problems from the exams

## Autumn 2008 5b:

Assume that each year 30% of owners of cars with two-wheel drive (2WD) change to a car with four-wheel drive (4WD), whilst 10% of owners of cars with 4WD change to a car with 2WD. The total number of cars is constant, and each car owner has only one car.

Given that 25% of car owners have 4WD now, what percentage of car owners will have 4WD drive in ten years' time?

# Two problems from the exams II

## Spring 2011 6b:

There are two places in Trondheim with bicycles that can be hired for free: Gløshaugen (G) and Torget (T). The bicycles can be hired from early in the morning and must be returned to one of the places the same evening.

It is found that of the bicycles hired from G, 80% are returned to G and 20% to T. Of the bicycles hired from T, 30% are returned to G and 70% to T. We assume that this pattern is constant, that all bicycles are hired out each morning, and that no bicycles are stolen.

In the long term, what proportion of the bicycles will be at Gløshaugen each morning?

# Similarities

- The population is divided into a finite set of mutually exclusive states.
- The system evolves in discrete time intervals and in each interval the individuals can change state.
- An individual changes state according to a set list of probabilities that depends only on the current state and is independent of time.

This situation often occurs when we model (dynamical) systems in the natural sciences!

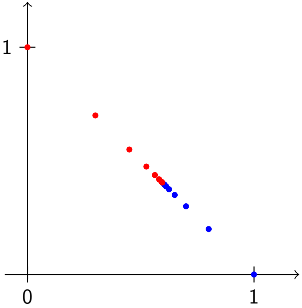
We call such systems **Markov chains**.

## 15.3 Spring 2011 6 b
## Example for a Markov chain

$$P = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix},$$

$$\overrightarrow{x}_{i+1} = P\overrightarrow{x_i}, \quad i = 1, 2, 3, \dots$$

Two initial states $\overrightarrow{x}_0$: $\overrightarrow{e}_1$, $\overrightarrow{e}_2$




| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $\overrightarrow{e}_1$ | $\begin{bmatrix} .8 \\ .2 \end{bmatrix}$ | $\begin{bmatrix} .7 \\ .3 \end{bmatrix}$ | $\begin{bmatrix} .65 \\ .35 \end{bmatrix}$ | $\begin{bmatrix} .625 \\ .375 \end{bmatrix}$ | $\begin{bmatrix} .6125 \\ .3875 \end{bmatrix}$ | $\begin{bmatrix} .6062 \\ .3938 \end{bmatrix}$ | $\begin{bmatrix} .6031 \\ .3969 \end{bmatrix}$ |
| $\overrightarrow{e}_2$ | $\begin{bmatrix} .3 \\ .7 \end{bmatrix}$ | $\begin{bmatrix} .45 \\ .55 \end{bmatrix}$ | $\begin{bmatrix} .525 \\ .475 \end{bmatrix}$ | $\begin{bmatrix} .5625 \\ .4375 \end{bmatrix}$ | $\begin{bmatrix} .5812 \\ .4188 \end{bmatrix}$ | $\begin{bmatrix} .5906 \\ .4094 \end{bmatrix}$ | $\begin{bmatrix} .5953 \\ .4047 \end{bmatrix}$ |

Apparently for both initial values the system runs towards $\begin{bmatrix} .6 \\ .4 \end{bmatrix}$.

# Questions connected to Markov chains

- What happens in the next (/after finitely many) steps?
- What is the long term behaviour of the system?

# The long term behaviour of Markov chains

A stochastic matrix $P$ is called **regular** if there is some $k \in \mathbb{N}$ such that $P^k$ has only strictly positive entries.

## Examples

$$P = \begin{bmatrix} .5 & .25 & .25 \\ 0 & .25 & .25 \\ .5 & .5 & .5 \end{bmatrix} \text{ is regular since } P^2 = \begin{bmatrix} .375 & .3125 & .3125 \\ .125 & .1875 & .1875 \\ .5 & .5 & .5 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ is not regular}$$

# The long term behaviour of Markov chains II

**Theorem**

If $P$ is a regular $n \times n$ stochastic matrix, then $P$ has a unique steady-state vector $\overrightarrow{q}$.
For any intial state $\overrightarrow{x}_0$ the Markov chain $\{x_k\}_{k \in \mathbb{N}_0}$ with $\overrightarrow{x}_{k+1} = P\overrightarrow{x}_k$ converges to $\overrightarrow{q}$ as $k \to \infty$.

To find a steady-state vector:

- Check if the stochastic matrix $P$ is regular
- Solve $(P - I)\overrightarrow{x} = \overrightarrow{0}$,
  find a probability vector solving the equation.
  Do **not** try to compute $P^k \overrightarrow{x}_0$ for $k \to \infty$